§9.2 (Part 2): Formulating Integer Programming Problems

1.] Capital Budgeting Problem: An investment firm is considering four investments. Investment 1 will yield an NPV of \$16,000; investment 2, an NPV of \$22,000; investment 3, an NPV of \$12,000; and investment 4, an NPV of \$8,000. Each investment requires a certain cash outflow at the present time: investment 1, \$5,000; investment 2, \$7,000; investment 3, \$4,000; and investment 4, \$3,000. Currently, \$14,000 is available for investment. Formulate an IP that will maximize NPV while meeting the following requirements:

- They can only invest in at most two investments.
- If they invest in investment 2, they cannot invest in investment 4.
- If they invest in investment 2, they must invest in investment 1 as well.

Decision Variables: let $x_i = \begin{cases} 1 & \text{Firm chooses investment } i \\ 0 & \text{Firm does not choose investment } i \end{cases}$

Obj. Fun: Maximize $z = 16x_1 + 22x_2 + 12x_3 + 8x_4$

Constraints:

(Budget) $5x_1 + 7x_2 + 4x_3 + 3x_4 \le 14$

(Bullet 1) $x_1 + x_2 + x_3 + x_4 \le 4$

(Bullet 2) $x_2 + x_4 \le 1$

(Bullet 3) $-x_1 + x_2 \le 0$

Bullet 3: If $x_2 = 1$, then $x_1 = 1$

$x_2 \le x_1$

Case 1: If $x_2 = 1$, then $x_1 \ge 1 \Rightarrow x_1 = 1$.

Case 2: If $x_2 = 0$, then $x_1 \ge 0$ which allows x_1 to be 0 or 1.

2.] Either-Or Constraints: Dorian Auto is considering manufacturing three types of cars: compact, midsize, and large. The resources for, and the profits yielded by, each type of car are shown in the table below. Currently, 6,000 tons of steel and 60,000 hours of labor are available. For production of a type of car to be economically feasible, at least 1,000 cars of each type must be produced. Formulate an IP to maximize Dorian's profit.

	Car Type		
Resource	Compact	Midsize	Large
Steel required	1.5 tons	3 tons	5 tons
Labor required	30 hours	25 hours	40 hours
Profit yielded	\$2,000	\$3,000	\$4,000

Decision Variables:

x_i = # of cars of type i produced

$y_i = \begin{cases} 1 & \text{if type } i \text{ is produced} \\ 0 & \text{otherwise} \end{cases}$

Objective Function: Maximize $z = 2000x_1 + 3000x_2 + 4000x_4$

Constraints: $x_i, y_i \ge 0$, $y_i \in \{0,1\}$.

(Steel) $1.5x_1 + 3x_2 + 5x_3 \le 6000$

(Labor) $30x_1 + 25x_2 + 40x_3 \le 60\,000$

(Minimum Production of Compact) $\begin{cases} x_1 \le My_1 \\ 1000 - x_1 \le M(1-y_1) \end{cases}$

(Minimum Production of Midsize) $\begin{cases} x_2 \le My_2 \\ 1000 - x_2 \le M(1-y_2) \end{cases}$

(Minimum Production of Large) $\begin{cases} x_3 \le My_3 \\ 1000 - x_3 \le M(1-y_3) \end{cases}$

*Constructing the minimum production constraints:

Either $x_1 \le 0$ or $x_1 \ge 1000$

Define $f := -x_1$ and $g := 1000 - x_1$

Then "either $f \ge 0$ or $g \ge 0$" becomes

$\begin{cases} f \le My_1 \\ g \le M(1-y_1) \end{cases} \Rightarrow \begin{cases} -x_1 \le My_1 \\ 1000 - x_1 \le M(1-y_1) \end{cases}$

3.] IF-THEN CONSTRAINTS: Coach Bobby Knight is trying to choose the starting lineup for the basketball team. The team consists of seven players who have been rated (on a scale of 1 to 3) according to their ball-handling, shooting, rebounding, and defensive abilities. The positions that each player is allowed to play and the player's abilities are listed in the table below.

Player	Position	Handling	Shooting	Rebounding	Defense
1	G	3	3	1	3
2	C	2	1	3	2
3	G-F	2	3	2	2
4	F-C	1	3	3	1
5	G-F	3	3	3	3
6	F-C	3	1	2	3
7	G-F	3	2	2	3

The five-player starting lineup must satisfy the following restrictions:

- At least ~~4~~ 3 members must be able to play guard, at least 2 members must be able to play forward, and at least 1 member must be able to play center.
- The average ball-handling, shooting, and rebounding level of the starting lineup must be at least 2.
- If player 3 starts, then player 6 cannot start.
- If player 1 starts, then players 4 and 5 must both start.
- Either player 2 or player 3 must start.

Given these constraints, Coach Knight wants to maximize the total defensive ability of the starting team. Formulate an IP that will help him choose his starting lineup.

Decision Variables: $x_i = \begin{cases} 1 & \text{if player } i \text{ starts} \\ 0 & \text{otherwise} \end{cases}$

Objective Function: Maximize $z = 3x_1 + 2x_2 + 2x_3 + x_4 + 3x_5 + 3x_6 + 3x_7$

Constraints: $x_i = \{0,1\}$ for all i.

(5-player Team) $x_1+x_2+x_3+x_4+x_5+x_6+x_7 = 5$

(Guards) $x_1+x_3+x_5+x_7 \geq 3$

(Forwards) $x_3+x_4+x_5+x_6+x_7 \geq 2$

(Centers) $x_2+x_4+x_6 \geq 1$

(Handling) $x_1 - x_4 + x_5 + x_6 + x_7 \geq 0$

(Shooting) $x_1 - x_2 + x_3 + x_4 + x_5 - x_6 \geq 0$

(Rebounding) $-x_1 + x_2 + x_4 + x_5 - x_6 - x_7 \geq 0$

(Bullet 3) $x_3 + x_6 \leq 1$

(Bullet 5) $x_2 + x_3 \geq 1$

(Bullet 4) $\begin{cases} -x_4 - x_5 + 2 \leq My \\ x_1 \leq M(1-y) \end{cases}$

*Constructing the Bullet 4 constraints:

If $x_1 > 0$, then $x_4 + x_5 \geq 2$

Define $f := x_1$ and $g := x_4 + x_5$

Then "if $f > 0$, then $g \geq 0$" becomes

$\begin{cases} -g \leq My_1 \\ f \leq M(1-y_1) \end{cases} \Rightarrow \begin{cases} -x_4 - x_5 + 2 \leq My \\ x_1 \leq M(1-y) \end{cases}$