§6.8: Shadow Prices

1.] A company manufactures two products (1 and 2). Each unit of product 1 can be sold for \$15, and each unit of product 2 can be sold for \$25. Each product requires raw material and two types of labor (skilled and unskilled) (see the table below). Currently, the company has available 100 hours of skilled labor, 70 hours of unskilled labor, and 30 units of raw material. Because of marketing considerations, at least 3 units of product 2 must be produced. The relevant LP is given below.

	Product	
Resource	1	2
Skilled Labor (hours)	3	4
Unskilled Labor (hours)	2	3
Raw Material (units)	1	2

Maximize: $z = 15x_1 + 25x_2$

Subject to:
$$\begin{aligned} 3x_1 + 4x_2 &\le 100 \\ 2x_1 + 3x_2 &\le 70 \\ x_1 + 2x_2 &\le 30 \\ x_2 &\ge 3 \\ x_1, x_2 &\ge 0 \end{aligned}$$

Suppose Row 0 of the optimal tableau is

Row	Basic	z	x_1	x_2	s_1	s_2	s_3	e_4	a_4	RHS
0	z	1	0	0	0	0	15	5	$(M-5)$	435

a.) How much would the company be willing to pay for an extra unit of raw material?

The shadow price of raw material is $y_3 = 15$.
Hence, they'd be willing to pay \$15 for 1 unit of raw material.

b.) Assuming the current basis remains optimal, what would the company's revenue be if 35 units of raw material were available?

$\Delta b = 5$ $\quad$ (New z_{opt}) = (Old z_{opt}) + $\Delta b_3 y_3$

$= 435 + (5)(15) = \boxed{\$510}$

c.) Assuming the current basis remains optimal, what would the company's revenue be if 80 hours of skilled labor were available?

$\Delta b_1 = -20$ $\quad$ (New z_{opt}) = (Old z_{opt}) + $\Delta b_1 y_1$

$= 435 + (-20)(0) = \boxed{\$435}$

Stays the same, skilled labor is abundant.

d.) Assuming the current basis remains optimal, what would the company's revenue be if at least 5 units of product 2 were required?

$\Delta b_4 = 2$ $\quad$ (New z_{opt}) = (Old z_{opt}) + $\Delta b_4 y_4$

$= 435 + (2)(-5) = \boxed{\$425}$

2.] The ToyCo company uses three operations to assemble three types of toys – trains, trucks, and cars. The daily available times for the three operations are 430, 460, and 420 minutes, respectively, and the revenues per unit of toy train, truck, and car are \$3, \$2, and \$5, respectively. The assembly times per train at the three operations are 1, 3, and 1 minutes, respectively. The corresponding times per train and per car are $(2, 0, 4)$ and $(1, 2, 0)$, respectively. Letting x_1, x_2, and x_3 be the daily number of units assembled of trains, trucks, and cars, respectively, the associated LP and optimal tableau are given as:

Maximize: $z = 3x_1 + 2x_2 + 5x_3$

Subject to:
$$\begin{aligned} x_1 + 2x_2 + x_3 &\leq 430 \\ 3x_1 \quad + 2x_3 &\leq 460 \\ x_1 + 4x_2 \quad &\leq 420 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

Row	Basic	z	x_1	x_2	x_3	s_1	s_2	s_3	RHS
0	z	1	4	0	0	1	2	0	1350
1	x_2	0	$-\frac{1}{4}$	1	0	$\frac{1}{2}$	$-\frac{1}{4}$	0	100
2	x_3	0	$\frac{3}{2}$	0	1	0	$\frac{1}{2}$	0	230
3	s_3	0	2	0	0	-2	1	1	20

a.) What is the feasibility range of operation 1?

Reading the entries of B^{-1}, we must have

$X_1 = 100 + \frac{1}{2}\Delta b_1 - \frac{1}{4}\Delta b_2 \geq 0$

$X_2 = 230 + \frac{1}{2}\Delta b_2 \geq 0$

$X_3 = 20 - 2\Delta b_1 + \Delta b_2 + \Delta b_3 \geq 0$

let $\Delta b_2 = \Delta b_3 = 0$, then

$100 + \frac{1}{2}\Delta b_1 \geq 0$

$230 \geq 0$ $\Rightarrow$

$20 - 2\Delta b_1 \geq 0$

$\Delta b_1 \geq -200$

$\Delta b_1 \leq 10$

$\Rightarrow$ Feasibility Range: $\boxed{230 \leq b_1 \leq 440}$

b.) Suppose the availabilities of operations 1, 2, and 3 are changed to 438, 500, and 410 minutes, respectively. Use the set of simultaneous conditions to show that the current basis remains feasible, and determine the new optimal value using the shadow prices.

Using the conditions above, we set $\Delta b_1 = 8$, $\Delta b_2 = 40$, and $\Delta b_3 = -10$, then the new optimal solution under these changes would be

$X_1 = 100 + \frac{1}{2}(8) - \frac{1}{4}(40) = 100 + 4 - 10 = 94 \geq 0$

$X_2 = 230 + \frac{1}{2}(40) = 230 + 20 = 250 \geq 0$

$X_3 = 20 - 2(8) + 40 - 10 = 20 - 16 + 30 = 34 \geq 0$

Since they're all non-negative, this basis is still optimal.