§6.7 (Part 2): The Duality Theorem

1.] Consider the following LP:

$$\text{Maximize:} \quad z = 2x_1 + 4x_2 + 4x_3 - 3x_4$$

$$\begin{aligned} \text{Subject to:} \quad x_1 + x_2 + x_3 &= 4 \\ x_1 + 4x_2 + x_4 &= 8 \\ x_1, x_2, x_3, x_4 &\geq 0 \end{aligned}$$

$\vec{c} = [2\ 4\ 4\ -3]$ $\vec{b} = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$ $A = \begin{array}{c} \\ \end{array}\begin{array}{cccc} \vec{a}_1 & & & \\ \end{array}\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 4 & 0 & 1 \end{bmatrix}$

a.) Identify, from the primal problem, the vectors $\boldsymbol{c}$ and $\boldsymbol{b}$, and the matrix A.

$$\vec{c} = [2\ 4\ 4\ -3]$$

$$\vec{b} = \begin{bmatrix} 4 \\ 8 \end{bmatrix} \qquad A = \begin{array}{c} \begin{array}{cccc} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 & \vec{a}_4 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 4 & 0 & 1 \end{bmatrix} \end{array}$$

b.) Suppose x_2 and x_3 comprise the optimal basis to the primal problem. Determine $\boldsymbol{c}_{BV}$, B, and B^{-1}. Then, show that it is optimal by computing $\bar{c}_j$ for the non-basic variables.

$$\vec{c}_B = [4\ 4], \quad B = \begin{bmatrix} 1 & 1 \\ 4 & 0 \end{bmatrix} \longrightarrow B^{-1} = \frac{1}{-4}\begin{bmatrix} 0 & -1 \\ -4 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1/4 \\ 1 & -1/4 \end{bmatrix}$$

Reduced Costs (Row 0 Coeffs) for non-basic variables:

- $$\bar{c}_1 = \vec{c}_B B^{-1} \vec{a}_1 - c_1 = [4\ 4]\begin{bmatrix} 0 & 1/4 \\ 1 & -1/4 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} - 2$$
$$= [4\ 4]\begin{bmatrix} 1/4 \\ 3/4 \end{bmatrix} - 2 = (1+3) - 2 = 2 > 0 \ ✓$$

- $$\bar{c}_4 = \vec{c}_B B^{-1} \vec{a}_4 - c_4 = [4\ 4]\begin{bmatrix} 0 & 1/4 \\ 1 & -1/4 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} - (-3)$$
$$= [4\ 4]\begin{bmatrix} 1/4 \\ -1/4 \end{bmatrix} + 3 = (1-1) + 3 = 3 > 0 \ ✓$$

Since both reduced costs are positive, we know the basis $\{x_2, x_3\}$ is optimal.

c.) Write the dual problem.

Dual LP: Minimize $w = 4y_1 + 8y_2$

Subject to

$$\begin{aligned} y_1 + y_2 &\geq 2 \\ y_1 + 4y_2 &\geq 4 \\ y_1 \quad &\geq 4 \\ y_2 &\geq -3 \\ y_1, y_2 &\text{ urs} \end{aligned}$$

d.) Find the optimal solutions to the primal and dual, and verify the Dual Theorem.

Solution to Primal LP:

- $\vec{X}_B = B^{-1}\vec{b} \longrightarrow \begin{bmatrix} x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 & 1/4 \\ 1 & -1/4 \end{bmatrix}\begin{bmatrix} 4 \\ 8 \end{bmatrix} = \begin{bmatrix} 0(4) + \frac{1}{4}(8) \\ 1(4) - \frac{1}{4}(8) \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$

- $z_{opt} = \vec{C}_B \vec{X}_B = [4 \ \ 4]\begin{bmatrix} 2 \\ 2 \end{bmatrix} = 8 + 8 = 16$

$\Rightarrow \boxed{x_1 = 0, \ x_2 = 2, \ x_3 = 2, \ x_4 = 0 \ ; \ z_{opt} = 16}$

Solution to Dual LP:

- $\vec{y} = \vec{C}_B B^{-1} \longrightarrow [y_1 \ \ y_2] = [4 \ \ 4]\begin{bmatrix} 0 & 1/4 \\ 1 & -1/4 \end{bmatrix} = [4 \ \ 0]$

- $w_{opt} = \vec{y}\vec{b} = [4 \ \ 0]\begin{bmatrix} 4 \\ 8 \end{bmatrix} = 16 + 0 = 16$

$\Rightarrow \boxed{y_1 = 4 \quad y_2 = 0 \ ; \ w_{opt} = 16}$