§6.7 (part 1): The Duality Theorem

1.] Prove: If the primal is unbounded, then the dual problem is infeasible.

Proof: look at the contrapositive:

If the dual is feasible, then the primal is bounded.

Let $\vec{y}$ be a feasible solution to the dual. Let $\vec{x}$ be any feasible solution to the primal. By Weak Duality,

$$z = \vec{c}^T\vec{x} \leq \vec{b}^T\vec{y} = w$$

so the primal is bounded. ∎

2.] For the following LP,

$$\text{Maximize:} \quad z = -x_1 + 5x_2$$

$$\begin{aligned} \text{Subject to:} \quad x_1 + 2x_2 &\leq 0.5 \\ -x_1 + 3x_2 &\leq 0.5 \\ x_1, x_2 &\geq 0 \end{aligned}$$

the Row 0 of the optimal tableau is

Row	Basic	z	x_1	x_2	s_1	s_2	RHS
0	z	1	0	0	0.4	1.4	??

What is the optimal z-value of the given LP?

Dual Solution: $y_1 = .4$, $y_2 = 1.4$

Dual obj. fun: $w = .5y_1 + .5y_2$

Dual Optimum: $w_{opt} = .5(.4) + .5(1.4) = .2 + .7 = .9$

Primal Optimum: $z_{opt} = w_{opt} = \boxed{0.9}$

3.] Consider the following LP:

$$\text{Maximize:} \quad z = -2x_1 - x_2 + x_3$$

$$\begin{aligned} \text{Subject to:} \quad x_1 + x_2 + x_3 &\leq 3 \\ x_2 + x_3 &\geq 2 \\ x_1 \qquad + x_3 &= 1 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

The Row 0 of the optimal tableau is

Row	Basic	z	x_1	x_2	x_3	s_1	e_2	a_2	a_3	RHS
0	z	1	4	0	0	0	1	$(M-1)$	$(M+2)$	0

z_{opt}

What is the optimal solution to the dual LP? Verify that the optimal objective value function for the dual is the same as the primal.

Dual Optimal Solution: $y_1 = 0$

$y_2 = -1$

$y_3 = M + 2 - M = 2$

Dual Minimum: $w = 3y_1 + 2y_2 + y_3$

$w_{opt} = 3(0) + 2(-1) + 2 = \boxed{0} = z_{opt}$