§6.6: Economic Interpretation of the Dual Problem

1.] Recall the Reddy Mikks problem, where a company produces both interior and exterior paints from two raw materials, M_1 and M_2. The first two constraints are raw material constraints. The third constraint makes sure the daily demand for interior paint cannot exceed that for exterior paint by more than 1 ton. Lastly, the maximum daily demand for interior paint is 2 tons. The objective function is measured in \$1000s. The maximization LP is formulated as follows:

Primal LP:

Maximize: $z = 5x_1 + 4x_2$

Subject to:
$$6x_1 + 4x_2 \le 24$$
$$x_1 + 2x_2 \le 6$$
$$-x_1 + x_2 \le 1$$
$$x_2 \le 2$$
$$x_1, x_2 \ge 0$$

Dual LP:

Minimize $w = 24y_1 + 6y_2 + y_3 + 2y_4$

Subject to:
$$6y_1 + y_2 - y_3 \ge 5$$
$$4y_1 + 2y_2 + y_3 + y_4 \ge 4$$
$$y_1, y_2, y_3, y_4 \ge 0$$

a.) Find the dual problem and interpret the dual variables below.

y_1 = cost/ton of Raw Material M_1

y_2 = cost/ton of Raw Material M_2

y_3 = cost/ton of difference in demand

y_4 = cost/ton of demand of interior paint.

b.) Note that $x_1 = x_2 = 1$ is a feasible solution for the primal and that $y_1 = y_2 = y_3 = y_4 = 1$ is a feasible solution for the dual. Show that $z < w$.

Primal:

$Z = 5(1) + 4(1)$

$\Rightarrow Z = 9$

Dual:

$W = 24(1) + 6(1) + 1(1) + 2(2)$

$\Rightarrow W = 33$

Weak Duality

$Z < W$

$9 < 33$ ✓

c.) Interpret the expression $\sum_{i=1}^{m} a_{ij} y_i - c_j$. equivalent to $\bar{c}_j$!

- $\sum_{i=1}^{m} a_{ij} y_i$ = "total cost/worth of resources used to produce product j" (Dual interpretation)

  a_{ij}: Resource i/Product j $\quad$ y_i: Cost/Resource i

- c_j = "selling price of product j" (Primal interpretation)
- If $\sum_{i=1}^{m} a_{ij} y_i - c_j < 0$, then the variable x_j should be entered into basis!

d.) The optimal tableau for the Reddy Mikks problem is given below. Identify the optimal solution for the primal and deduce the optimal ~~value~~ solution of the dual from this tableau. Then, show $z = w$ at optimality.

Row	Basic	z	x_1	x_2	s_1	s_2	s_3	s_4	RHS
0	z	1	0	0	$\frac{3}{4}$	$\frac{1}{2}$	0	0	21
1	x_1	0	1	0	$\frac{1}{4}$	$-\frac{1}{2}$	0	0	3
2	x_2	0	0	1	$-\frac{1}{8}$	$\frac{3}{4}$	0	0	$\frac{3}{2}$
3	s_3	0	0	0	$\frac{9}{24}$	$-\frac{5}{4}$	1	0	$\frac{5}{2}$
4	s_4	0	0	0	$\frac{1}{8}$	$-\frac{3}{4}$	0	1	$\frac{1}{2}$

Primal:

Optimal solution is $\underline{x_1 = 3}$, $\underline{x_2 = 1.5}$, $S_3 = 2.5$, $S_4 = 0.5$

$z_{opt} = 21$

Dual: Primal is a normal max problem, hence

Optimal solution is

$y_1 = 0.75, \quad y_2 = 0.5, \quad y_3 = 0, \quad y_4 = 0$

$W_{opt} = 24(0.75) + 6(0.5) + 1(0) + 2(0)$

$\Rightarrow W_{opt} = 18 + 3 + 0 + 0$

$\Rightarrow \boxed{W_{opt} = 21}$ ← Same as z_{opt}.

Note: $y_1 = 0.75 \rightarrow$ "M_1 is worth \$750/ton"

$y_2 = 0.5 \rightarrow$ "M_2 is worth \$500/ton"

$y_3 = y_4 = 0 \rightarrow$ "Demand constraints are abundant"