§5.1 (PART 2): LIMIT OF A RIEMANN SUM

1.] Express the right Riemann Sum of $f(x) = x^2$ on $[0, 9]$ using n rectangles in summation notation. The answer to this question is an expression involving n only.

• Width: $\Delta x = \frac{b-a}{n} = \frac{9-0}{n} = \frac{9}{n}$

∘ Heights: $x_k^* = a + k\Delta x = 0 + k \cdot \frac{9}{n} = \frac{9k}{n}$

$f(x_k^*) = f\left(\frac{9k}{n}\right) = \left(\frac{9k}{n}\right)^2 = \frac{81k^2}{n^2}$

• Riemann Sum: $R_n = \sum_{k=1}^{n} f(x_k^*)\Delta x$

$= \sum_{k=1}^{n} \left(\frac{81k^2}{n^2}\right)\left(\frac{9k}{n}\right)$

$= \sum_{k=1}^{n} \frac{729k^2}{n^3}$

$= \frac{729}{n^3} \sum_{k=1}^{n} k^2$

$= \frac{729}{n^3}\left(\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}\right)$

$= \frac{729}{3} + \frac{729}{2n} + \frac{729}{6n^2}$

$$\boxed{R_n = 243 + \frac{729}{2n} + \frac{243}{2n^2}}$$

2.] Using your expression from part 1, find the right-hand Riemann Sum with $n = 9$ rectangles.

$$R_9 = 243 + \frac{729}{2(9)} + \frac{243}{2(9)^2}$$

$$R_9 = 243 + 40.5 + 1.5$$

$$\boxed{R_9 = 285}$$

3.] Using your expression from part 1, find the right-hand Riemann Sum with $n = 18$ rectangles.

$$R_{18} = 243 + \frac{729}{2(18)} + \frac{243}{2(18)^2}$$

$$R_{18} = 243 + 20.25 + 0.375$$

$$\boxed{R_{18} = 263.625}$$

4.] Using your expression from part 1, find the right-hand Riemann Sum with $n = 180$ rectangles.

$$R_{180} = 243 + \frac{729}{2(180)} + \frac{243}{2(180)^2}$$

$$R_{18} = 243 + 20.25 + 0.375$$

$$\boxed{R_{180} = 245.0288}$$

5.] Find the exact value of the area under the curve $f(x) = x^2$ on $[0, 9]$ by taking $n \to \infty$.

$$\boxed{\text{Exact Area}} = \lim_{n\to\infty} R_n$$

$$= \lim_{n\to\infty} 243 + \frac{729}{2n} + \frac{243}{2n^2}$$

$$= 243 + 0 + 0$$

$$\boxed{\lim_{n\to\infty} R_n = 243}$$

Total Distance = 243 miles