§4.6: Optimization Problems

1.] What two positive real numbers with a product of of 50 have the smallest possible sum?

Objective Function: Minimize Sum, $S = x+y$

Constraint: $xy = 50$

$\Rightarrow y = \frac{50}{x}$

$S = x + \frac{50}{x}$

$\Rightarrow S(x) = x + 50x^{-1}$

$\Rightarrow S'(x) = 1 - 50x^{-2}$

$1 - 50x^{-2} = 0$

$\Rightarrow 1 = 50x^{-2}$

$\Rightarrow x^2 = 50$

$\Rightarrow x = \sqrt{50}$

$\Rightarrow \boxed{x = 5\sqrt{2}}$

$S''(x) = 100x^{-3}$

$S''(x) = \frac{100}{x^3}$

$S''(5\sqrt{2}) = \frac{100}{(5\sqrt{2})^3} > 0$

minimum

$y = \frac{50}{5\sqrt{2}} = \frac{25}{\sqrt{2}} \Rightarrow \boxed{y = 5\sqrt{2}}$

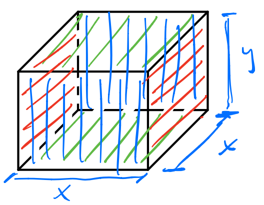
2.] Of all boxes with a square base and a volume of 100 m^3, which one has the smallest minimum surface area?

Objective Function: Minimize Surface Area, $S = 4xy + 2x^2$

Constraint: $x \cdot x \cdot y = 100$

$\Rightarrow x^2 y = 100$

$\Rightarrow y = \frac{100}{x^2}$



$S = 4x\left(\frac{100}{x^2}\right) + 2x^2$

$\Rightarrow S(x) = \frac{400}{x} + 2x^2$

$\Rightarrow S(x) = 400x^{-1} + 2x^2$

$\Rightarrow S'(x) = -400x^{-2} + 4x$ $\Rightarrow S''(x) = 800x^{-3} + 4$

$\Rightarrow 0 = -400x^{-2} + 4x$

$\Rightarrow 400x^{-2} = 4x$

$\Rightarrow 100 = x^3$

$\Rightarrow \boxed{x = \sqrt[3]{100}}$

minimum!

$\Rightarrow S''(x) = \frac{800}{x^3} + 4$

$\Rightarrow S''(\sqrt[3]{100}) = \frac{800}{100} + 4$

$= 12 > 0$

$\Rightarrow y = \frac{100}{(\sqrt[3]{100})^2}$

$\Rightarrow y = \frac{100}{100^{2/3}}$

$\boxed{y = \sqrt[3]{100}}$

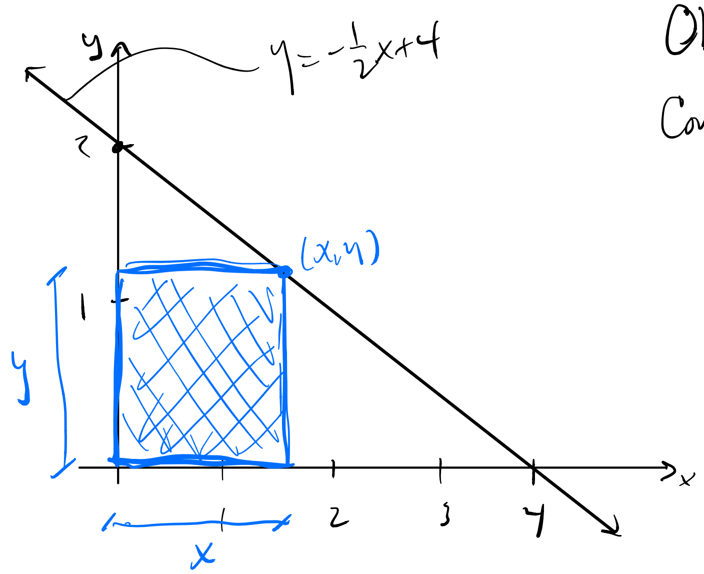
Surface Area $= \underbrace{2xy + 2xy}_{} + 2x^2 = 4xy + 2x^2$

Front/Back: $xy + xy = 2xy$

Left/Right: $xy + xy = 2xy$

Top/Bottom: $xx + xx = 2x^2$

3.] A vertex of a rectangle is at the origin; the opposite vertex sits in the first quadrant and on the line $2y + x = 4$. Find the dimensions that maximize the area of such a rectangle.



Objective Function: maximize Area, $A = xy$

Constraint: $y = -\frac{1}{2}x + 2$

$A = x\left(-\frac{1}{2}x + 2\right)$

$\Rightarrow A(x) = -\frac{1}{2}x^2 + 2x$

$\Rightarrow A'(x) = -x + 2$

$\Rightarrow 0 = -x + 2$

$\Rightarrow \boxed{x = 2}$

max!

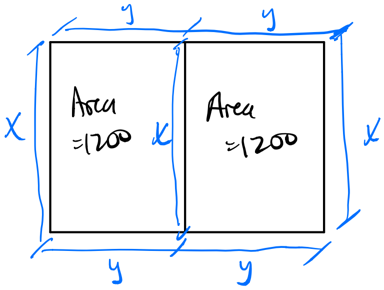
$A''(x) = -1 < 0$

$\Rightarrow y = -\frac{1}{2}(2) + 2$

$\Rightarrow y = -1 + 2$

$\Rightarrow \boxed{y = 1}$

4.] A farmer has a large rectangular pen that he wishes to subdivide into two adjoining rectangular pens of identical areas. If each pen is to have an area of 1200 square feet, what dimensions will minimize the cost of fencing.



Objective Function: Minimize Fencing, $F = 3x+4y$

Constraint: $xy = 1200$

$\Rightarrow y = \frac{1200}{x}$

$F = 3x + 4\left(\frac{1200}{x}\right)$

$\Rightarrow F(x) = 3x + \frac{4800}{x}$

$\Rightarrow F(x) = 3x + 4800x^{-1}$

$\Rightarrow F'(x) = 3 - 4800x^{-2}$ $\rightarrow F''(x) = 9600x^{-3}$

$F''(x) = \frac{9600}{x^3}$

$\Rightarrow 0 = 3 - 4800x^{-2}$

$\Rightarrow 4800x^{-2} = 3$

$\Rightarrow 4800 = 3x^2$

$\Rightarrow x^2 = 1600$

$\Rightarrow \boxed{x = 40}$

$F''(40) = \frac{9600}{40^3} > 0$

minimum!

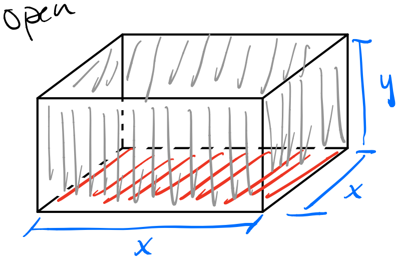
$\Rightarrow y = \frac{1200}{40}$

$\Rightarrow \boxed{y = 30}$

Total Fencing $= x+x+x+y+y+y+y$
$= 3x+4y$

5.] You work for a company that makes jewelry boxes. Your boss tells you that each jewelry box must have a square base and an open top and that you can spend \$3.75 on the materials for each box. The people in production tell you that the material for the sides of the box costs 2 cents per square inch while the reinforced material for the base of the box costs 5 cents per square inch. What is the largest volume jewelry box that you can make and still stay within budget?

open



- Sides
  - Surface Area $= 4xy$
  - Cost $= .02 \cdot 4xy = .08xy$
- Bottom
  - Surface Area $= x^2$
  - Cost $= .05x^2$

Total Cost:
$C = .08xy + .05x^2$

Objective Function: Maximize Volume, $V = x^2y$

Constraint: Cost $= 3.75$

$\Rightarrow .08xy + .05x^2 = 3.75$

$\Rightarrow .08xy = 3.75 - .05x^2$

$\Rightarrow y = \frac{3.75}{.08x} - \frac{5}{8}x$

$\Rightarrow V = x^2\left(\frac{3.75}{.08x} - \frac{5}{8}x\right)$

$\Rightarrow V(x) = \frac{375}{8}x - \frac{5}{8}x^3$

$\Rightarrow V'(x) = \frac{375}{8} - \frac{15}{8}x^2$

$\Rightarrow 0 = \frac{375}{8} - \frac{15}{8}x^2$

$\Rightarrow \frac{15}{8}x^2 = \frac{375}{8}$

$\Rightarrow x^2 = \frac{8}{15} \cdot \frac{375}{8}$

$\Rightarrow x^2 = 25$

$\Rightarrow \boxed{x = 5}$

$V''(x) = -\frac{30}{8}x$

$\Rightarrow V''(5) = -\frac{30}{8}(5) < 0$

Max!

$y = \frac{375}{8(5)} - \frac{5}{8}(5)$

$\Rightarrow y = \frac{375}{40} - \frac{25}{8} \Rightarrow y = \frac{375-125}{40} \Rightarrow y = \frac{250}{4} \Rightarrow \boxed{y = 6.25}$