§4.4: L'Hôpital's Rule

1.] Determine the indeterminate form of the limits below, then evaluate them using L'Hôpital's Rule.

a.) $\lim_{x\to 5} \dfrac{4x-20}{2x-10}$ $\left[=\dfrac{\infty}{\infty}\right]$

L'H $= \lim_{x\to 5} \dfrac{4}{2} = \dfrac{4}{2} = \boxed{2}$

OLD WAY:

$\lim_{x\to 5} \dfrac{4x-20}{2x-10} = \lim_{x\to 5} \dfrac{4\cancel{(x-5)}}{2\cancel{(x-5)}} = \lim_{x\to 5} \dfrac{4}{2} = \boxed{2}$

b.) $\lim_{x\to 0} \dfrac{3^x-1}{x}$ $\left[=\dfrac{3^0-1}{0} = \dfrac{1-1}{0} = \dfrac{0}{0}\right]$

L'H $\lim_{x\to 0} \dfrac{\ln(3)\cdot 3^x}{1} = \lim_{x\to 0} \ln(3)\cdot 3^x = \ln(3)\cdot 3^0 = \boxed{\ln(3)}$

c.) $\lim_{x\to 0} \dfrac{\sqrt{9+3x}-3}{x}$ $\left[=\dfrac{\sqrt{9+3(0)}-3}{0} = \dfrac{\sqrt{9}-3}{0} = \dfrac{3-3}{0} = \dfrac{0}{0}\right]$

L'H $\lim_{x\to 0} \dfrac{\frac{1}{2}(9+3x)^{-1/2}\cdot 3}{1} = \lim_{x\to 0} \dfrac{3}{2\sqrt{9+3x}} = \dfrac{3}{2\sqrt{9+3(0)}} = \dfrac{3}{2\sqrt{9}} = \dfrac{3}{2\cdot 3} = \boxed{\dfrac{1}{2}}$

2.] Determine the indeterminate form of the limits below, then evaluate them using L'Hôpital's Rule.

a.) $\lim_{x\to\infty} \dfrac{3x^4+x}{6x^4+12}$ $\left[=\dfrac{\infty+\infty}{\infty+12} = \dfrac{\infty}{\infty}\right]$

L'H $\lim_{x\to\infty} \dfrac{12x^3+1}{24x^3}$ $\left[=\dfrac{\infty}{\infty}\right]$

L'H $\lim_{x\to\infty} \dfrac{36\cancel{x^2}}{72\cancel{x^2}} = \lim_{x\to\infty} \dfrac{36}{72} = \dfrac{36}{72} = \boxed{\dfrac{1}{2}}$

b.) $\lim_{x\to\infty} \dfrac{x^2+\ln(x)}{3x^2+2x}$ $\left[=\dfrac{\infty+\infty}{\infty+\infty} = \dfrac{\infty}{\infty}\right]$

L'H $\lim_{x\to\infty} \dfrac{2x+\frac{1}{x}}{6x+2}$ $\left[=\dfrac{\infty+0}{\infty+2} = \dfrac{\infty}{\infty}\right]$

L'H $\lim_{x\to\infty} \dfrac{2-\frac{1}{x^2}}{6} = \dfrac{2-0}{6} = \dfrac{2}{6} = \boxed{\dfrac{1}{3}}$

3.] Determine the indeterminate form of the limits below, then evaluate them using L'Hôpital's Rule.

a.) $\lim_{x\to 0^+} x\ln(x)$ $\left[= 0\cdot(-\infty)\right]$

$$= \lim_{x\to0^+} \frac{\ln(x)}{\frac{1}{x}} \quad \left[= \frac{-\infty}{\infty}\right]$$

L'H

$$\lim_{x\to0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}}$$

$$= \lim_{x\to0^+} -\frac{x^2}{x} = -\lim_{x\to0^+} x = -0 = \boxed{0}$$

b.) $\lim_{x\to 0^+} \cot(x) - \frac{1}{x}$

$$= \lim_{x\to0^+} \frac{\cos(x)}{\sin(x)} - \frac{1}{x} \quad \left[= \frac{1}{0^+} - \frac{1}{0^+} = \infty - \infty\right]$$

$$= \lim_{x\to0^+} \frac{x\cos(x) - \sin(x)}{x\sin(x)} \quad \left[= \frac{0\cdot\cos(0) - \sin(0)}{0\cdot\sin(0)} = \frac{0}{0}\right]$$

L'H

$$\lim_{x\to0^+} \frac{(1)\cancel{\cos(x)} + (x)(-\sin(x)) - \cancel{\cos(x)}}{(1)\sin(x) + x\cos(x)}$$

$$= \lim_{x\to0^+} \frac{-x\sin(x)}{\sin(x) + x\cos(x)} \quad \left[= \frac{0\cdot\sin(0)}{\sin(0) + 0\cdot\cos(0)} = \frac{0}{0}\right]$$

L'H

$$\lim_{x\to0^+} \frac{(-1)\sin(x) + (-x)\cos(x)}{\cos(x) + (1)\cos(x) + x(-\sin(x))}$$

$$= \lim_{x\to0^+} \frac{-\sin(x) - x\cdot\cos(x)}{2\cos(x) - x\sin(x)}$$

$$= \frac{-\sin(0) - 0\cdot\cos(0)}{2\cos(0) - 0\cdot\sin(0)} = \frac{0}{2} = \boxed{0}$$

4.] Determine the indeterminate form of the limits below, then evaluate them using L'Hôpital's Rule.

a.) $\lim_{x\to 0^+} x^{2x}$ $\left[= 0^{2\cdot 0} = 0^0\right]$

$$= \lim_{x\to0^+} e^{\ln(x^{2x})}$$

$$= \lim_{x\to0^+} e^{2x\ln(x)}$$

$$= e^{\lim_{x\to0^+} 2x\ln(x)} = e^0 = \boxed{1}$$

$$\lim_{x\to0^+} 2x\ln(x)$$

$$= 2\underbrace{\lim_{x\to0^+} x\ln(x)}_{\text{Proble 3a above}} \quad \left[= 0\cdot(-\infty)\right]$$

$$= 2\cdot 0$$

$$= 0$$

b.) $\lim_{x\to\infty} \left(1 + \frac{1}{x}\right)^x$ $\left[= (1+0)^\infty = 1^\infty\right]$

$$= \lim_{x\to\infty} e^{\ln\left(1+\frac{1}{x}\right)^x}$$

$$= \lim_{x\to\infty} e^{x\ln\left(1+\frac{1}{x}\right)}$$

$$= e^{\lim_{x\to\infty} x\ln\left(1+\frac{1}{x}\right)}$$

$$= e^1 = \boxed{e}$$

$$\lim_{x\to\infty} x\ln\left(1+\frac{1}{x}\right) \quad \left[= \infty\cdot\ln(1) = \infty\cdot 0\right]$$

$$= \lim_{x\to\infty} \frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}} \quad \left[= \frac{0}{0}\right]$$

$$= \lim_{x\to\infty} \frac{\frac{1}{1+\frac{1}{x}}\cdot\cancel{\left(-\frac{1}{x^2}\right)}}{\cancel{\left(-\frac{1}{x^2}\right)}}$$

$$= \lim_{x\to\infty} \frac{1}{1+\frac{1}{x}}$$

$$= \frac{1}{1+0}$$

$$= 1$$