§4.3 (Part 2): Second Derivative Test

1.] Determine the intervals of concavity of the function $f(x) = 3x^5 - 30x^4 + 80x^3 + 100$. Identify all inflection points.

Candidate Inflection Pts:

$$f'(x) = 15x^4 - 120x^3 + 240x^2$$

$$f''(x) = 60x^3 - 360x^2 + 480x$$

$$\Rightarrow f''(x) = 60x(x^2 - 6x + 8)$$

$$\Rightarrow \boxed{f''(x) = 60x(x-2)(x-4)}$$

$$60x(x-2)(x-4) = 0$$

$$60x = 0 \quad x-2 = 0 \quad x-4 = 0$$

$$x = 0,\ x = 2,\ x = 4$$

Inflection pts: $i_1 = 0$, $i_2 = 2$, $i_3 = 4$

Concave up on $(0,2) \cup (4,\infty)$

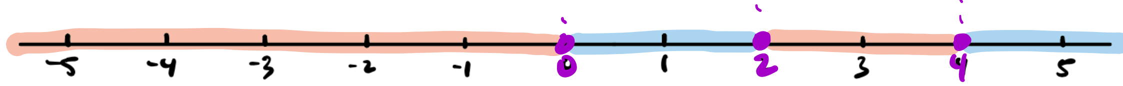
Concave down on $(-\infty,0) \cup (2,4)$

$f''(-1) = (-)(-)(-) < 0$

$f''(1) = (+)(-)(-)$ $f''(3) = (+)(+)(-)$ $f''(5) = (+)(+)(+)$

Concave Down CU CD CU

$f''(x)$



2.] Locate the critical points of the function $f(x) = x^2e^{-x}$. Then use the Second Derivative Test to determine whether the critical points correspond to local maxima, local minima, or neither.

Critical Pts: $f'(x) = 2xe^{-x} + e^{-x}(-1)x^2$

$$f'(x) = xe^{-x}(2-x)$$

$$xe^{-x}(2-x) = 0$$

$\boxed{x=0}$ $e^{-x} = 0$ N/A $2-x = 0$ $\boxed{x=2}$

$\boxed{c_1 = 0,\ c_2 = 2}$

2nd Derivative:

$$f''(x) = (x)'e^{-x}(2-x) + x(e^{-x})'(2-x) + xe^{-x}(2-x)'$$

$$f''(x) = e^{-x}(2-x) - xe^{-x}(2-x) - xe^{-x}$$

2nd Derivative Test: $f''(0) = e^0(2-0) - \cancel{(0)e^0(2-0)} - \cancel{(0)e^0} = 2 > 0 \rightarrow$ $\boxed{\text{Local min at } x=0}$

$f''(2) = e^{-2}\cancel{(2-2)} - 2\cancel{e^{-2}(2-2)} - 2e^{-2} = -2e^{-2} = \frac{-2}{e^2} < 0 \rightarrow$ $\boxed{\text{Local max at } x=2}$

3.] Determine the intervals of monotonicity and concavity of the function $f(x) = x^4 + 8x^3 - 270x^2 + 1$. Use the first or second derivative test to determine all local extreme values.

Monotonicity

$f'(x) = 4x^3 + 24x^2 - 540x$

$f'(x) = 4x(x^2 + 6x - 135)$

$f'(x) = 4x(x-9)(x+15)$

$\boxed{c_1 = -15,\ c_2 = 0,\ c_3 = 9}$

Critical Points

Concavity

$f''(x) = 12x^2 + 48x - 540$

$f''(x) = 12(x^2 + 4x - 45)$

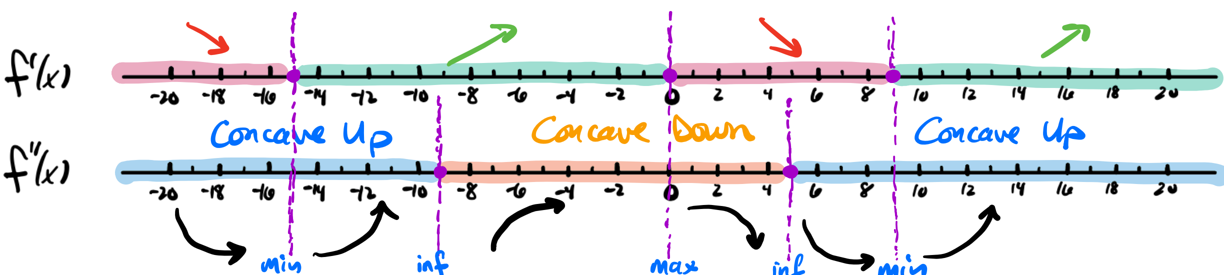
$f''(x) = 12(x+9)(x-5)$

$\boxed{i_1 = -9,\ i_2 = 5}$

inflection pts

Summary

- Increasing on $(-15, 0) \cup (9, \infty)$
- Decreasing on $(-\infty, -15) \cup (0, 9)$
- Concave up on $(-\infty, -9) \cup (5, \infty)$
- Concave down on $(-9, 5)$
- Local min at $x = -15$, $x = 9$
- Local max at $x = 0$.



4.] Determine the intervals of monotonicity and concavity of the function $f(x) = 27(x-2)^3(x+2)$. Use the First or Second Derivative Test to determine all local extreme values.

Monotonicity

$f'(x) = 81(x-2)^2(x+2) + 27(x-2)^3(1)$

$f'(x) = 27(x-2)^2[3(x+2) + (x-2)]$

$f'(x) = 27(x-2)^2(4x+4)$

$f'(x) = 108(x-2)^2(x+1)$

$\boxed{c_1 = -1,\ c_2 = 2}$

critical pts

Concavity

$f''(x) = 216(x-2)(x+1) + 108(x-2)^2(1)$

$f''(x) = 108(x-2)[2(x+1) + (x-2)]$

$f''(x) = 108(x-2)(3x)$

$f''(x) = 324x(x-2)$

$\boxed{i_1 = 0,\ i_2 = 2}$ Inflection pts.