§4.2: Mean Value Theorem

1.] Verify that $f(x) = x^3 - x^2 - 6x + 2$ satisfies the hypothesis of Rolle's theorem for the interval $[0, 3]$ and then find all c that satisfy the conclusion.

Hypothesis:

1.) f is continuous on $[0,3]$ because it is a polynomial. ✓

2.) $f'(x) = 3x^2 - 2x - 6$. Since f' is a polynomial, it is continuous on $(0,3)$ i.e. f is differentiable. ✓

3.) $f(0) = 0^3 - 0^2 - 6(0) + 2 = 2$

$f(3) = 3^3 - 3^2 - 6(3) + 2 = 27 - 9 - 18 + 2 = 2$ } same ✓

Conclusion: There exists a $c \in (0,3)$ such that

$$f'(c) = 0$$

Quadratic Formula

$$\Rightarrow 3c^2 - 2c - 6 = 0$$

$$\Rightarrow c = \frac{-(-2)}{2(3)} \pm \frac{\sqrt{(-2)^2 - 4(3)(-6)}}{2(3)}$$

$$\Rightarrow c = \frac{1}{3} \pm \frac{\sqrt{4+72}}{6}$$

$$\Rightarrow c = \frac{1}{3} \pm \frac{\sqrt{76}}{6}$$

$$\Rightarrow c = \frac{1}{3} \pm \frac{\sqrt{4 \cdot 19}}{6}$$

$$\Rightarrow c = \frac{1}{3} \pm \frac{\sqrt{19}}{3}$$

$$\Rightarrow \boxed{c = \frac{1}{3} + \frac{\sqrt{19}}{3}}, \quad \cancel{c = \frac{1}{3} - \frac{\sqrt{19}}{3}}$$

outside of interval

2.] Let $f(x) = \dfrac{3}{(x-1)^2}$. Show that $f(0) = f(2)$ but that there is no value $c \in (0, 2)$ such that $f'(c) = 0$. Why does this not contradict Rolle's theorem?

Hypothesis:

1.) $f(x) = \frac{3}{(x-1)^2}$ has a vertical asymptote at $x = 1$. Not continuous on $[0,2]$ ✗

2.) $f(x) = \frac{3}{(x-1)^2}$ is not differentiable on $(0,2)$ since $f(1) = \text{DNE}$ ✗

3.) $f(0) = \frac{3}{(0-1)^2} = \frac{3}{(-1)^2} = 3$

$f(2) = \frac{3}{(2-1)^2} = \frac{3}{(1)^2} = 3$ } same ✓

Conclusion: Since $f(x) = \frac{3}{(x-1)^2}$ does not satisfy all three conditions in the hypothesis, we cannot conclude that there exists a c inside $(0,2)$ such that $f'(c) = 0$. This is not a contradiction to the theorem, we simply cannot apply the theorem.

3.] Verify that $f(x) = x^3 - 3x + 2$ satisfies the hypotheses of the Mean Value Theorem on $[-2, 2]$ and then find all c that satisfy the conclusion.

Hypothesis:

1.) $f(x) = x^3 - 3x + 2$ is a polynomial and so f is continuous on $[-2,2]$. ✓

2.) $f'(x) = 3x^2 - 3$ is also a polynomial and so f' is continuous on $(-2,2)$, which means f is differentiable on $(-2,2)$. ✓

Conclusion: There exists a c inside $(-2,2)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$\Rightarrow 3c^2 - 3 = \frac{f(2) - f(-2)}{2 - (-2)}$$

$$\Rightarrow 3c^2 - 3 = \frac{[(2)^3 - 3(2) + 2] - [(-2)^3 - 3(-2) + 2]}{4}$$

$$\Rightarrow 3(c^2 - 1) = \frac{(8 - 6 + 2) - (-8 + 6 + 2)}{4}$$

$$\Rightarrow 3(c^2 - 1) = \frac{4}{4}$$

$$\Rightarrow 3(c^2 - 1) = 1$$

$$\Rightarrow c^2 - 1 = \frac{1}{3}$$

$$\Rightarrow c^2 = \frac{4}{3}$$

$$\Rightarrow c = \pm \frac{\sqrt{4}}{\sqrt{3}}$$

$$\Rightarrow \boxed{c_1 = \frac{-2}{\sqrt{3}}, \quad c_2 = \frac{2}{\sqrt{3}}}$$

Both values are inside $(-2,2)$

4.] Determine if the function $f(x) = \dfrac{x+1}{2x-5}$ satisfies the hypothesis of the Mean Value Theorem on the interval $[-1, 2]$. If it does, find all values of c that satisfy the conclusion of the theorem.

Hypothesis:

1.) $f(x)$ is a rational function with domain $(-\infty, \frac{5}{2}) \cup (\frac{5}{2}, \infty)$. Since $f(x)$ is defined on $[-1,2]$, the function is continuous. ✓

2.) $f'(x) = \dfrac{(2x-5)(1) - (x+1)(2)}{(2x-5)^2} = \dfrac{2x-5-2x-2}{(2x-5)^2} = \dfrac{-7}{(2x-5)^2}$

$f'(x)$ is a rational function with domain $(-\infty, \frac{5}{2}) \cup (\frac{5}{2}, \infty)$. Since $f(x)$ is defined on $(-1,2)$, the derivative is continuous, i.e. $f(x)$ is differentiable on $(-1,2)$. ✓

Conclusion: There exists a value c inside $(-1,2)$ such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

$$\frac{-7}{(2c-5)^2} = \frac{f(2)-f(-1)}{2-(-1)}$$

$$f(2) = \frac{2+1}{2(2)-5} = \frac{3}{-1} = -3 \qquad f(-1) = \frac{-1+1}{2(-1)-5} = 0$$

$$\Rightarrow \frac{-7}{(2c-5)^2} = \frac{-3-0}{3}$$

$$\Rightarrow \frac{-7}{(2c-5)^2} = -1$$

$$\Rightarrow -7 = -(2c-5)^2$$

$$\Rightarrow (2c-5)^2 = 7$$

$$\Rightarrow 2c-5 = \pm\sqrt{7}$$

$$\Rightarrow 2c = 5 \pm \sqrt{7}$$

$$\Rightarrow c = \frac{5}{2} \pm \frac{\sqrt{7}}{2}$$

$$\Rightarrow \boxed{c_1 = \frac{5}{2} - \frac{\sqrt{7}}{2}}, \quad \cancel{c_2 = \frac{5}{2} + \frac{\sqrt{7}}{2}}$$

not in interval

5.] Law enforcement has been known to issue speeding tickets to drivers who pass between successive EZ pass booths in too short of a time interval. Assume EZ pass booths A and B are 100 miles apart. Use the mean value theorem to demonstrate that a driver who passes booth A at 1 PM and booth B at 2 PM was necessarily speeding at some time between the two booths. What assumptions are you making about the driver's position function?

- Essentially, the driver travelled 100 miles in one hour. Thus, the average velocity between EZ Pass booths is 100 miles/hr. By the MVT, at some point between the booths, the driver must have been travelling at an instaneous velocity of 100 miles/hr and so he/she was absolutely speeding.

- We're assuming the driver is in a car (not a teleportation device) and so we're assuming the car is continuous, i.e. not traveling a positive distance in 0 secs.