§3.6: Derivatives of Inverse Functions

1.] Differentiate the following functions:

a.) $f(x) = 2\ln(x)\cos(x)$ Product Rule.

$$f'(x) = (2\ln(x))'(\cos(x)) + (2\ln(x))(\cos(x))'$$

$$\Rightarrow f'(x) = \frac{2}{x}\cos(x) + 2\ln(x)(-\sin(x))$$

$$\Rightarrow \boxed{f'(x) = \frac{2\cos(x)}{x} - 2\ln(x)\sin(x)}$$

b.) $g(x) = \log_2(2x^2 + 5)$

$$g'(x) = \frac{1}{\ln(2)\cdot(2x^2+5)} \cdot 4x$$

Chain Rule

$$\Rightarrow g'(x) = \frac{4x}{\ln(2)\cdot(2x^2+5)}$$

2.] Prove that $\dfrac{d}{dx}(\arctan(x)) = \dfrac{1}{1+x^2}$

Let $y = \arctan(x)$, then we know $\tan(y) = x$ by the definition of $\arctan(x)$. Then using implicit differentiation, we have:

$$\frac{d}{dx}(\tan(y)) = \frac{d}{dx}(x)$$

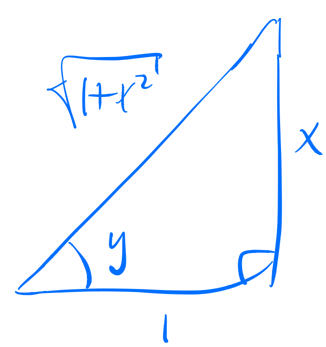
$$\Rightarrow \sec^2(y)\cdot\frac{dy}{dx} = 1$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sec^2(y)}$$

$$\Rightarrow \frac{dy}{dx} = \cos^2(y)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{1+x^2}$$

$$\tan(y) = x = \frac{x}{1} = \frac{\text{Opp}}{\text{Adj}}$$



$$\cos(y) = \frac{1}{\sqrt{1+x^2}}$$

$$\cos^2(y) = \frac{1}{1+x^2}$$

Hence, $\dfrac{d}{dx}(\arctan(x)) = \dfrac{1}{1+x^2}$

3.] Differentiate the following functions:

a.) $f(x) = x\arctan(x)$

$f'(x) = (x)'(\arctan(x)) + (x)(\arctan(x))'$

$\Rightarrow f'(x) = \arctan(x) + x\left(\frac{1}{1+x^2}\right)$

$\Rightarrow \boxed{f'(x) = \arctan(x) + \frac{x}{1+x^2}}$

b.) $g(x) = \sin\left(\arccos(2x)\right)$

$g'(x) = \cos(\arccos(2x)) \cdot \frac{-1}{\sqrt{1-(2x)^2}} \cdot 2$

$\Rightarrow g'(x) = 2x \cdot \frac{-2}{\sqrt{1-4x^2}}$

$\Rightarrow \boxed{g'(x) = \frac{-4x}{\sqrt{1-4x^2}}}$

c.) $h(x) = \cot^{-1}\left(\frac{1}{x^2+1}\right)$

$h(x) = \operatorname{arccot}\left(\frac{1}{x^2+1}\right)$

$h'(x) = -\frac{1}{1+\left(\frac{1}{x^2+1}\right)^2} \cdot \left(\frac{(x^2+1)(0)-(1)(2x)}{(x^2+1)^2}\right)$

$h'(x) = \frac{(-1)(-2x)}{\frac{(x^2+1)^2+1}{\cancel{(x^2+1)^2}} \cdot \cancel{(x^2+1)^2}} \Rightarrow \boxed{h'(x) = \frac{2x}{(x^2+1)^2+1}}$

d.) $k(x) = \ln\left(\tan^{-1}(x)\right)$

$k(x) = \ln(\arctan(x))$

$\Rightarrow k'(x) = \frac{1}{\arctan(x)} \cdot \frac{1}{1+x^2}$

$\Rightarrow \boxed{k'(x) = \frac{1}{\arctan(x)(1+x^2)}}$

4.] Find the equation of the tangent line to the curve defined by $y = \arccos(x^2)$ at the point $\left(\frac{1}{\sqrt{2}}, \frac{\pi}{3}\right)$.

Point: $(x_1, y_1) = \left(\frac{1}{\sqrt{2}}, \frac{\pi}{3}\right)$

Slope: $m = f'\left(\frac{1}{\sqrt{2}}\right) = -\frac{2}{3}\sqrt{6}$

Tangent Line: $y - y_1 = m(x - x_1)$

$\Rightarrow y - \frac{\pi}{3} = -\frac{2}{3}\sqrt{6}\left(x - \frac{1}{\sqrt{2}}\right)$

$\Rightarrow y - \frac{\pi}{3} = -\frac{2}{3}\sqrt{6}x + \frac{2\sqrt{6}}{3\sqrt{2}}$

$\Rightarrow y = -\frac{2}{3}\sqrt{6}x + \frac{\sqrt{12}}{3} + \frac{\pi}{3}$

$\Rightarrow \boxed{y = -\frac{2}{3}\sqrt{6}x + \frac{2\sqrt{3}+\pi}{3}}$

$f'(x) = -\frac{1}{\sqrt{1-(x^2)^2}} \cdot (2x)$

$= \frac{-2x}{\sqrt{1-x^4}}$

$f'\left(\frac{1}{\sqrt{2}}\right) = \frac{-2(1/\sqrt{2})}{\sqrt{1-\left(\frac{1}{\sqrt{2}}\right)^2}}$

$= \frac{-\sqrt{2}}{\sqrt{1-\frac{1}{4}}}$

$= \frac{-\sqrt{2}}{\sqrt{\frac{3}{4}}}$

$= \frac{-\sqrt{2}}{\frac{\sqrt{3}}{2}}$

$= -\frac{2\sqrt{2}}{\sqrt{3}}$

$= -\frac{2}{3}\sqrt{6}$