§3.5: Implicit Differentiation

1.] Verify that the point $(3, 2)$ lies on the curve defined by the relation

$$x^2 + xy - y^3 = 7.$$

Find the equation of the tangent line to the curve at the point $(3, 2)$.

Verify: $3^2 + (3)(2) - 2^3 = 9 + 6 - 8 = 9 - 2 = 7$ ✓

Point: $(x_1, y_1) = (3, 2)$

Slope: $m = \left.\frac{dy}{dx}\right|_{(x,y)=(3,2)} = -\frac{7}{9}$

$$\frac{d}{dx}\left(x^2 + xy - y^3\right) = \frac{d}{dx}(7)$$

$$\Rightarrow 2x + \frac{d}{dx}(x)\cdot y + x\cdot\frac{dy}{dx} - 3y^2\cdot\frac{dy}{dx} = 0$$

$$\Rightarrow 2x + y + x\frac{dy}{dx} - 3y^2\frac{dy}{dx} = 0$$

$$\Rightarrow x\frac{dy}{dx} - 3y^2\frac{dy}{dx} = -2x - y$$

$$\Rightarrow (x - 3y^2)\frac{dy}{dx} = -2x - y$$

$$\Rightarrow \frac{dy}{dx} = \frac{-2x - y}{x - 3y^2}$$

$$\Rightarrow \left.\frac{dy}{dx}\right|_{(3,2)} = \frac{-2(3) - 2}{3 - 3(2)^2} = \frac{-8}{3 - 12} = \frac{8}{9}$$

Line: $y - y_1 = m(x - x_1)$

$$\Rightarrow y - 2 = \frac{8}{9}(x - 3)$$

$$\Rightarrow y - 2 = \frac{8}{9}x - \frac{8}{3}$$

$$y = \frac{8}{9}x - \frac{8}{3} + \frac{6}{3}$$

$$\boxed{y = \frac{8}{9}x - \frac{2}{3}}$$

2.] Find $\frac{dy}{dx}$ for the curve defined by the following relation:

$$\sin(xy) = x^2 + y.$$

$$\frac{d}{dx}(\sin(xy)) = \frac{d}{dx}(x^2 + y)$$

$$\Rightarrow \cos(xy)\cdot\frac{d}{dx}(xy) = 2x + \frac{dy}{dx}$$

$$\Rightarrow \cos(xy)\cdot\left(\frac{d}{dx}(x)\cdot y + x\frac{d}{dx}(y)\right) = 2x + \frac{dy}{dx}$$

$$\Rightarrow \cos(xy)\cdot\left(y + x\frac{dy}{dx}\right) = 2x + \frac{dy}{dx}$$

$$\Rightarrow y\cos(xy) + x\cos(xy)\cdot\frac{dy}{dx} = 2x + \frac{dy}{dx}$$

$$\Rightarrow x\cos(xy)\frac{dy}{dx} - \frac{dy}{dx} = 2x - y\cos(xy)$$

$$\frac{dy}{dx}\left(x\cos(xy) - 1\right) = 2y - y\cos(xy)$$

$$\Rightarrow \boxed{\frac{dy}{dx} = \frac{2y - y\cos(xy)}{x\cos(xy) - 1}}$$

3.] Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ for the ellipse defined by the following relation:

$$2x^2 + y^2 = 4.$$

First Derivative:

$$\frac{d}{dx}(2x^2) + \frac{d}{dx}(y^2) = \frac{d}{dx}(4)$$

$$\Rightarrow \quad 4x + 2y\frac{dy}{dx} = 0$$

$$\Rightarrow \quad 2y\frac{dy}{dx} = -4x$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-4x}{2y}$$

$$\Rightarrow \quad \boxed{\frac{dy}{dx} = -\frac{2x}{y}}$$

Second Derivative:

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(-\frac{2x}{y}\right)$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{(y)\frac{d}{dx}(-2x) - (-2x)\frac{d}{dx}(y)}{y^2}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{(y)(-2) + (2x)\frac{dy}{dx}}{y^2}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{1}{y^2}\left[-2y + 2x\left(\frac{-2x}{y}\right)\right]$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{1}{y^2}\left[-2y - \frac{4x^2}{y}\right]$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{1}{y^2}\left[\frac{-2y^2 - 4x^2}{y}\right]$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{-(2y^2 + 4x^2)}{y^3}$$

Note: from the original equation $4x^2 + 2y^2 = 4$!

$$\Rightarrow \quad \boxed{\frac{d^2y}{dx^2} = -\frac{4}{y^3}}$$