§3.3: Derivatives of Trig Functions

1.] Differentiate the following functions:

a.) $f(x) = e^x \cos(x)$

$f'(x) = (e^x)'(\cos(x)) + (e^x)(\cos(x))'$

$\Rightarrow f'(x) = e^x\cos(x) + e^x(-\sin(x))$

$\Rightarrow \boxed{f'(x) = e^x(\cos(x) - \sin(x))}$

b.) $g(x) = \dfrac{\sin(x)}{x^2 + e^x}$

$g'(x) = \dfrac{(x^2+e^x)(\sin(x))' - (\sin(x))(x^2+e^x)'}{(x^2+e^x)^2}$

$\Rightarrow g'(x) = \dfrac{(x^2+e^x)(\cos(x)) - (\sin(x))(2x+e^x)}{(x^2+e^x)^2}$

$\Rightarrow \boxed{g'(x) = \dfrac{e^x(\cos(x) - \sin(x)) + x^2\cos(x) - 2x\sin(x)}{(x^2+e^x)^2}}$

c.) $h(x) = \sin(x) - x\cos(x)$

$h'(x) = (\sin(x))' - [(x)'(\cos(x)) + (x)(\cos(x))']$

$\Rightarrow h'(x) = \cos(x) - [\cos(x) + x(-\sin(x))]$

$\Rightarrow h'(x) = \cos(x) - \cos(x) + x\sin(x)$

$\Rightarrow \boxed{h'(x) = x\sin(x)}$

d.) $k(x) = \dfrac{1 + \sin(x)}{1 - \sin(x)}$

$k'(x) = \dfrac{(1-\sin(x))(1+\sin(x))' - (1+\sin(x))(1-\sin(x))'}{(1-\sin(x))^2}$

$\Rightarrow k'(x) = \dfrac{(1-\sin(x))(\cos(x)) - (1+\sin(x))(-\cos(x))}{(1-\sin(x))^2}$

$\Rightarrow k'(x) = \dfrac{\cos(x) - \cancel{\sin(x)\cos(x)} + \cos(x) + \cancel{\sin(x)\cos(x)}}{(1-\sin(x))^2}$

$\Rightarrow \boxed{k'(x) = \dfrac{2\cos(x)}{(1-\sin(x))^2}}$

2.] Differentiate the following functions:

a.) $f(x) = \tan(x)$

$f(x) = \dfrac{\sin(x)}{\cos(x)}$

$\Rightarrow f'(x) = \dfrac{(\cos(x))(\sin(x))' - (\sin(x))(\cos(x))'}{\cos^2(x)}$

$\Rightarrow f'(x) = \dfrac{\cos(x)\cos(x) - \sin(x)(-\sin(x))}{\cos^2(x)}$

$\Rightarrow f'(x) = \dfrac{\cos^2(x) + \sin^2(x)}{\cos^2(x)}$

$\Rightarrow f'(x) = \dfrac{1}{\cos^2(x)}$

$\Rightarrow \boxed{f'(x) = \sec^2(x)}$

$\dfrac{d}{dx}(\tan(x)) = \sec^2(x)$

b.) $g(x) = \csc(x)$

$g(x) = \dfrac{1}{\sin(x)}$

$\Rightarrow g'(x) = \dfrac{(\sin(x))(1)' - (1)(\sin(x))'}{\sin^2(x)}$

$\Rightarrow g'(x) = \dfrac{(\sin(x))(0) - \cos(x)}{\sin^2(x)}$

$\Rightarrow g'(x) = \dfrac{-\cos(x)}{\sin^2(x)}$

$\Rightarrow g'(x) = -\dfrac{1}{\sin(x)} \cdot \dfrac{\cos(x)}{\sin(x)}$

$\Rightarrow \boxed{g'(x) = -\csc(x)\cot(x)}$

$\dfrac{d}{dx}(\csc(x)) = -\csc(x)\cot(x).$

3.] Differentiate the following functions:

a.) $f(x) = \cot^2(x)$

$f(x) = \cot(x)\cot(x)$

$\Rightarrow f'(x) = (\cot(x))'(\cot(x)) + (\cot(x))(\cot(x))'$

$\Rightarrow f'(x) = -\csc^2(x)\cot(x) + \cot(x)(-\csc^2(x))$

$\Rightarrow \boxed{f'(x) = -2\csc^2(x)\cot(x)}$

b.) $g(x) = 2^x \sec(x)$

$g'(x) = (2^x)'(\sec(x)) + (2^x)(\sec(x))'$

$\Rightarrow g'(x) = \ln(2)\cdot 2^x \cdot \sec(x) + 2^x \sec(x)\tan(x)$

$\Rightarrow \boxed{g'(x) = 2^x \sec(x)(\ln(2) + \tan(x))}$

4.] Evaluate the following limit:

$$\lim_{x\to 0} \frac{5}{x^2 \csc^2(x)}$$

$$= \lim_{x\to 0} \frac{5\sin^2(x)}{x^2}$$

$$= 5 \lim_{x\to 0} \frac{\sin(x)\sin(x)}{x \cdot x}$$

$$= 5\Big(\underbrace{\lim_{x\to 0} \frac{\sin(x)}{x}}_{=1}\Big)\Big(\underbrace{\lim_{x\to 0} \frac{\sin(x)}{x}}_{=1}\Big)$$

$$= 5 \cdot 1 \cdot 1$$

$$= \boxed{5}$$