§3.1: Differential Notation

1.] Let $f(x) = x^2 - x$. Assuming $y = f(x)$, use both the prime notation and the differential notations to evaluate $f'(x)$. Use the derivative to find $f'(2)$ and express your answer in both notations.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$
$$= \lim_{h\to 0} \frac{[(x+h)^2-(x+h)]-[x^2-x]}{h}$$
$$= \lim_{h\to 0} \frac{x^2+2xh+h^2-x-h-x^2+x}{h}$$
$$= \lim_{h\to 0} \frac{2xh+h^2-h}{h}$$
$$= \lim_{h\to 0} \frac{h(2x+h-1)}{h} = 2x+0-1 = 2x-1$$

Slope of TL at $x=2$: $f'(2) = 2(2)-1 = 3$

Newton:
$f(x) = x^2 - x$
$f'(x) = 2x-1$
$f'(2) = 3$

Leibniz:
$y = x^2 - x$
$\frac{dy}{dx} = 2x-1$
$\left.\frac{dy}{dx}\right|_{x=2} = 3$

2.] A tennis ball is hit vertically upward so that its position function is given by $s(t) = -16t^2 + 96t + 4$ feet above the ground at t seconds. Find the velocity and acceleration functions. What are the initial position and velocity values?

Note: Velocity, $v(t)$, is the first derivative of position i.e. $v(t) = s'(t)$
Acceleration, $a(t)$, is the first derivative of velocity i.e. $a(t) = v'(t)$. So that $a(t) = s''(t)$.

Let $s(t) = -16t^2 + 96t + 4$, then

$$v(t) = s'(t) = \lim_{h\to 0} \frac{s(t+h)-s(t)}{h}$$
$$= \lim_{h\to 0} \frac{[-16(t+h)^2+96(t+h)+4]-[-16t^2+96t+4]}{h}$$
$$= \lim_{h\to 0} \frac{-16t^2-32th-16h^2+96t+96h+4+16t^2-96t-4}{h}$$
$$= \lim_{h\to 0} \frac{-32th-16h^2+96h}{h}$$
$$= \lim_{h\to 0} \frac{h(-32t-16h+96)}{h}$$
$$= -32t-16(0)+96$$
$$\boxed{v(t) = -32t+96}$$

$$a(t) = v'(t) = \lim_{h\to 0} \frac{v(t+h)-v(t)}{h}$$
$$= \lim_{h\to 0} \frac{[-32(t+h)+96]-[-32t+96]}{h}$$
$$= \lim_{h\to 0} \frac{-32t-32h+96+32t-96}{h}$$
$$= \lim_{h\to 0} \frac{-32h}{h}$$
$$\boxed{a(t) = -32}$$

Position: $s(t) = -16t^2+96t+4$ ft
Velocity: $v(t) = -32t+96$ ft/sec
Acceleration: $a(t) = -32$ ft/sec^2 ← (gravity)

Initial Pos.: $s(0) = -16(0)^2+96(0)+4$
$\Rightarrow \boxed{s(0) = 4 \text{ ft}}$

Initial Vel.: $v(0) = -32(0)+96$
$\Rightarrow \boxed{v(0) = 96 \text{ ft/sec}}$

3.] Consider the piecewise function $f(x)$ given below. Notice from the graph that $f(x)$ is continuous at $x = 1$. Show that $f(x)$ is not differentiable $x = 1$ and sketch $f'(x)$ on the graph below.

$$f(x) = \begin{cases} x^2 & \text{if } x \leq 1 \\ -x+2 & \text{if } x > 1 \end{cases}$$

Continuity:

- $f(1) = 1^2 = 1$
- $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} x^2 = 1^2 = 1$
- $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} -x+2 = -1+2 = 1$ ✓
- f is continuous at $x=1$.

Differentiability:

Must check the "right" and "left" side derivatives.

$$f'(1) = \lim_{x\to 1} \frac{f(x)-f(1)}{x-1}$$

$$f'_-(1) = \lim_{x\to 1^-} \frac{f(x)-f(1)}{x-1} = \lim_{x\to 1^-} \frac{x^2-1}{x-1} = \lim_{x\to 1^-} \frac{(x+1)(x-1)}{x-1} = \lim_{x\to 1^-} x+1 = 1+1 = 2$$

$$f'_+(1) = \lim_{x\to 1^+} \frac{f(x)-f(1)}{x-1} = \lim_{x\to 1^+} \frac{(-x+2)-1}{x-1} = \lim_{x\to 1^+} \frac{-x+1}{x-1} = \lim_{x\to 1^+} \frac{-(x-1)}{x-1} = -1$$

not the same!

$f'(1) = \text{DNE}$

· Because the left and right side derivatives at $x=1$ are not equal, we say $f(x)$ is not differentiable at $x=1$

Derivative Function

For $x<1$: $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h\to 0} \frac{x^2+2xh+h^2-x^2}{h} = \lim_{h\to 0} \frac{2xh+h^2}{h} = \lim_{h\to 0} 2x+h$

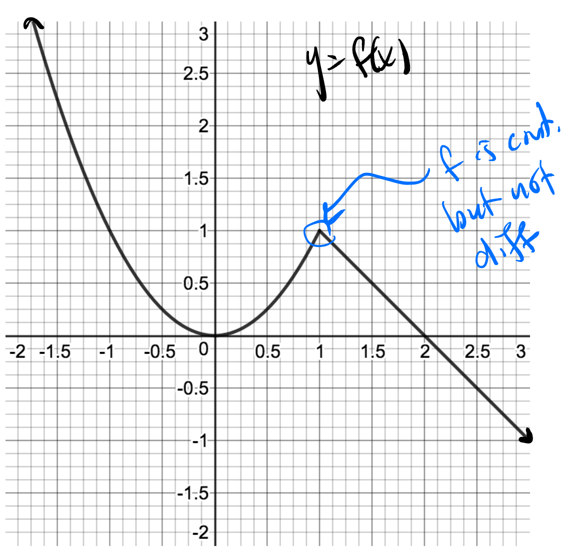
$f'(x) = 2x$ for $x<1$

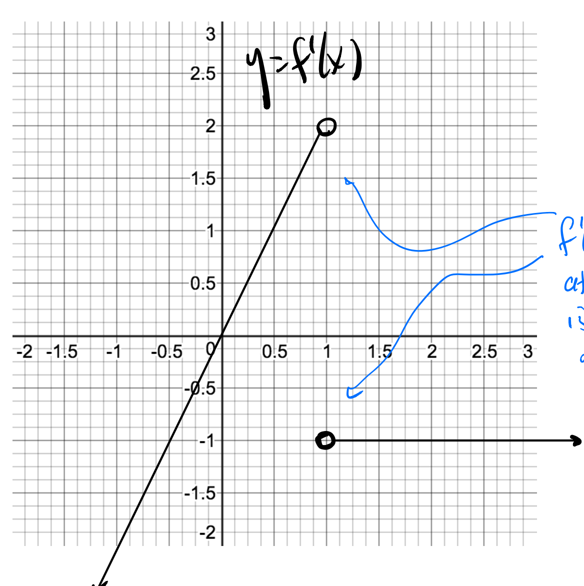
For $x>1$: $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{-(x+h)+2-(-x+2)}{h} = \lim_{h\to 0} \frac{-x-h+2+x-2}{h} = \lim_{h\to 0} -\frac{h}{h}$

$f'(x) = -1$ for $x>1$

$$f'(x) = \begin{cases} 2x & \text{if } x<1 \\ -1 & \text{if } x>1 \end{cases}$$

Note: $f'(1)$ doesn't exist





Note: A function is differentiable at c if its derivative function $f'(x)$ is continuous at c.