§2.6 (PART 2): THE DERIVATIVE FUNCTION

1.] Use the limit definition to find the derivative function of $f(x) = x^3 + x$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$f(x+h) = (x+h)^3 + (x+h)$$
$$= x^3 + 3x^2h + 3xh^2 + h^3 + x + h$$

$$= \lim_{h\to 0} \frac{(x^3 + 3x^2h + 3xh^2 + h^3 + x + h) - (x^3 + x)}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{x^3} + 3x^2h + 3xh^2 + h^3 + \cancel{x} + h - \cancel{x^3} - \cancel{x}}{h}$$

$$= \lim_{h\to 0} \frac{3x^2h + 3xh^2 + h^3 + h}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{h}(3x^2 + 3xh + h^2 + 1)}{\cancel{h}}$$

$$= \lim_{h\to 0} 3x^2 + 3xh + h^2 + 1$$

$$= 3x^2 + 3x(0) + (0)^2 + 1$$

$$= 3x^2 + 1$$

$$\boxed{f'(x) = 3x^2 + 1}$$

2.] Use the limit definition to find the derivative function of $f(x) = \sqrt{x}$ and use it to compute the equations of the tangent line to the graph of f at the points $(1, f(1))$ and $(2, f(2))$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}\left(\frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}\right)$$

$$= \lim_{h\to 0} \frac{(\cancel{x}+h) - \cancel{x}}{h(\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h\to 0} \frac{\cancel{h}}{\cancel{h}(\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h\to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}}$$

$$= \frac{1}{\sqrt{x+0} + \sqrt{x}}$$

$$= \frac{1}{\sqrt{x} + \sqrt{x}}$$

$$= \frac{1}{2\sqrt{x}}$$

Equation of TL at $(1, f(1))$:

Point: $(x_1, y_1) = (1, f(1)) = (1, \sqrt{1}) = (1, 1)$

Slope: $m = f'(1) = \frac{1}{2\sqrt{1}} = \frac{1}{2}$

Line: $y - y_1 = m(x - x_1)$

$\Rightarrow y - 1 = \frac{1}{2}(x - 1)$

$\Rightarrow y - 1 = \frac{1}{2}x - \frac{1}{2}$

$\Rightarrow \boxed{y = \frac{1}{2}x + \frac{1}{2}}$

Equation of TL at $(2, f(2))$:

Point: $(x_1, y_1) = (2, f(2)) = (2, \sqrt{2})$

Slope: $m = f'(2) = \frac{1}{2\sqrt{2}}$

Line: $y - y_1 = m(x - x_1)$

$\Rightarrow y - \sqrt{2} = \frac{1}{2\sqrt{2}}(x - 2)$

$\Rightarrow y - \sqrt{2} = \frac{1}{2\sqrt{2}}x - \frac{1}{\sqrt{2}}$

$\Rightarrow y = \frac{1}{2\sqrt{2}}x - \frac{1}{\sqrt{2}} + \sqrt{2}$

$\Rightarrow \boxed{y = \frac{1}{2\sqrt{2}}x + \frac{1}{\sqrt{2}}}$

3.] Use the limit definition to find the derivative function of $f(x) = \sqrt{x^2+1}$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{(x+h)^2+1} - \sqrt{x^2+1}}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{(x+h)^2+1} - \sqrt{x^2+1}}{h}\left(\frac{\sqrt{(x+h)^2+1} + \sqrt{x^2+1}}{\sqrt{(x+h)^2+1} + \sqrt{x^2+1}}\right)$$

$$= \lim_{h\to 0} \frac{(x+h)^2+1-(x^2+1)}{h\left(\sqrt{(x+h)^2+1} + \sqrt{x^2+1}\right)}$$

$$= \lim_{h\to 0} \frac{\cancel{x^2}+2xh+h^2\cancel{+1}\cancel{-x^2}\cancel{-1}}{h\left(\sqrt{(x+h)^2+1} + \sqrt{x^2+1}\right)}$$

$$= \lim_{h\to 0} \frac{\cancel{h}(2x+h)}{\cancel{h}\left(\sqrt{(x+h)^2+1} + \sqrt{x^2+1}\right)}$$

$$= \lim_{h\to 0} \frac{2x}{\left(\sqrt{(x+h)^2+1} + \sqrt{x^2+1}\right)}$$

$$= \frac{2x}{\sqrt{(x+0)^2+1} + \sqrt{x^2+1}}$$

$$= \frac{2x}{\sqrt{x^2+1} + \sqrt{x^2+1}}$$

$$= \frac{2x}{2\sqrt{x^2+1}}$$

$$\boxed{f'(x) = \frac{x}{\sqrt{x^2+1}}}$$