§2.6 (PART 1): THE DERIVATIVE AT A POINT

1.] A particle is moving along a straight line so that its distance from the starting position is given by $d(t) = 5t - \frac{1}{2}t^2$, where d is measured in feet and t in seconds. What is the particle's instantaneous velocity at $t = 2$ seconds?

$$d'(a) = \begin{bmatrix}\text{Instantaneous Velocity} \\ \text{at } t=a\end{bmatrix} = \lim_{b\to a} \frac{d(b)-d(a)}{b-a}$$

$$d'(2) = \begin{bmatrix}\text{Instantaneous Velocity} \\ \text{at } t=2\end{bmatrix} = \lim_{b\to 2} \frac{d(b)-d(2)}{b-2}$$

$$= \lim_{b\to 2} \frac{5b-\frac{1}{2}b^2-8}{b-2}$$

$$= \lim_{b\to 2} \frac{-\frac{1}{2}b^2+5b-8}{b-2}$$

$$= \lim_{b\to 2} \frac{-\frac{1}{2}(b^2-10b+16)}{b-2}$$

$$= \lim_{b\to 2} \frac{-\frac{1}{2}\cancel{(b-2)}(b-8)}{\cancel{b-2}}$$

$$= \lim_{b\to 2} -\frac{1}{2}(b-8)$$

$$= -\frac{1}{2}(2-8)$$

$$= -\frac{1}{2}(-6)$$

$$= 3$$

$$\boxed{d'(2) = 3 \text{ ft/sec}}$$

$d(t) = 5t - \frac{1}{2}t^2$

$\xrightarrow{t=b} d(b) = 5b - \frac{1}{2}b^2$

$\xrightarrow{t=2} d(2) = 5(2) - \frac{1}{2}(2)^2$
$= 10 - 2$
$= 8$

2.] Find the slope of the tangent line to the graph of $f(x) = \sqrt{x+1}$ at the point $(0, f(0))$.

$$f'(c) = \begin{bmatrix}\text{Slope of TL} \\ \text{at } x=c\end{bmatrix} = \lim_{h\to 0} \frac{f(c+h)-f(c)}{h}$$

$$f'(0) = \begin{bmatrix}\text{Slope of TL} \\ \text{at } x=0\end{bmatrix} = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{h+1}-1}{h}\left(\frac{\sqrt{h+1}+1}{\sqrt{h+1}+1}\right)$$

$$= \lim_{h\to 0} \frac{h+1-1}{h(\sqrt{h+1}+1)}$$

$$= \lim_{h\to 0} \frac{\cancel{h}}{\cancel{h}(\sqrt{h+1}+1)}$$

$$= \lim_{h\to 0} \frac{1}{\sqrt{h+1}+1}$$

$$= \frac{1}{\sqrt{0+1}+1}$$

$$= \frac{1}{1+1}$$

$$= \frac{1}{2}$$

$$\boxed{f'(0) = \frac{1}{2}}$$

$f(x) = \sqrt{x+1}$

$\xrightarrow{x=0} f(0) = \sqrt{0+1} = \sqrt{1} = 1$

$\xrightarrow{x=0+h} f(0+h) = \sqrt{0+h+1} = \sqrt{h+1}$

At the point $(0,1)$ on the graph of $y=\sqrt{x+1}$, the unique tangent line has slope $f'(0) = m = \frac{1}{2}$.

3.] Find the slope of the tangent line to the graph of $f(x) = \frac{1}{2}x^2 - 5x + 7$ at the point $(2, f(2))$. Use both forms of the limit definition to compute the slope of the tangent line. Use this information to determine the equation of the tangent line at that point.

For both definitions, we need to know $f(2)$:

$$f(2) = \frac{1}{2}(2)^2 - 5(2) + 7$$
$$= 2 - 10 + 7$$
$$= -1$$

we'll also need $f(2+h)$:

$$f(2+h) = \frac{1}{2}(2+h)^2 - 5(2+h) + 7$$
$$= \frac{1}{2}(4+4h+h^2) - 10 - 5h + 7$$
$$= 2 + 2h + \frac{1}{2}h^2 - 10 - 5h + 7$$
$$= \frac{1}{2}h^2 - 3h - 1$$

$$f'(c) = \lim_{x\to c} \frac{f(x) - f(c)}{x - c}$$
$$f'(2) = \lim_{x\to 2} \frac{f(x) - f(2)}{x-2}$$
$$= \lim_{x\to 2} \frac{(\frac{1}{2}x^2 - 5x + 7) - (-1)}{x-2}$$
$$= \lim_{x\to 2} \frac{\frac{1}{2}x^2 - 5x + 8}{x-2}$$
$$= \lim_{x\to 2} \frac{\frac{1}{2}(x^2 - 10x + 16)}{x-2}$$
$$= \lim_{x\to 2} \frac{1}{2}\frac{\cancel{(x-2)}(x-8)}{\cancel{x-2}}$$
$$= \lim_{x\to 2} \frac{1}{2}(x-8)$$
$$= \frac{1}{2}(2-8)$$
$$= \frac{1}{2}(-6)$$
$$\boxed{f'(2) = -3}$$

$$f'(c) = \lim_{h\to 0} \frac{f(c+h) - f(c)}{h}$$
$$f'(2) = \lim_{h\to 0} \frac{f(2+h) - f(2)}{h}$$
$$= \lim_{h\to 0} \frac{(\frac{1}{2}h^2 - 3h - 1) - (-1)}{h}$$
$$= \lim_{h\to 0} \frac{\frac{1}{2}h^2 - 3h - 1 + 1}{h}$$
$$= \lim_{h\to 0} \frac{\frac{1}{2}h^2 - 3h}{h}$$
$$= \lim_{h\to 0} \frac{\cancel{h}(\frac{1}{2}h - 3)}{\cancel{h}}$$
$$= \lim_{h\to 0} \frac{1}{2}h - 3$$
$$= \frac{1}{2}(0) - 3$$
$$\boxed{f'(2) = -3}$$

Equation of Tangent Line:

Slope: $m = f'(2) = -3$

Point: $(2, f(2)) = \underset{(x_1, y_1)}{(2, -1)}$

$$y - y_1 = m(x - x_1)$$
$$y - (-1) = -3(x-2)$$
$$y + 1 = -3x + 6$$
$$\boxed{y = -3x + 5}$$