§2.5 (PART 2): CONTINUITY

1.] Determine the interval on which the function $f(x) = \dfrac{1}{x^2-4}$ is continuous.

Rational functions are continuous on their domains so it is continuous on $(-\infty,-2)\cup(-2,2)\cup(2,\infty)$

2.] Determine if this piecewise function is continuous on the entire real number line.

$$f(x) = \begin{cases} \frac{2x}{2-x} & \text{if } x < 1 \\ x^2+3x & \text{if } 1 \le x \le 3 \\ \frac{x^2-5x+6}{3-x} & \text{if } x > 3 \end{cases}$$

Continuous on $(-\infty,1)\cup(1,3)\cup(3,\infty)$

Check Continuity at $x=1$:

1.) $f(1) = 1^2+3(1) = 4$ ✓

2.) $\lim\limits_{x\to 1^-} \frac{2x}{2-x} = \frac{2(1)}{2-1} = \frac{2}{1} = 2$ ✗

$\lim\limits_{x\to 1^+} x^2+3x = 1^2+3(1) = 4$

$\Rightarrow$ Not Continuous at $x=1$, but it is right-continuous.

Check Continuity at $x=3$:

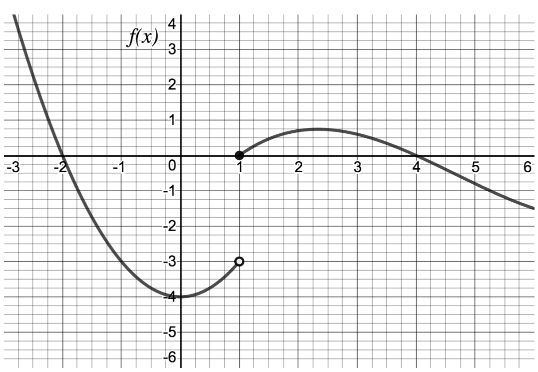
1.) $f(3) = 3^2+3(3) = 9+9 = 18$ ✓

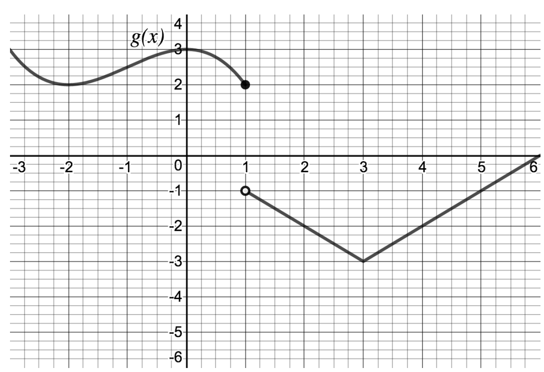
2.) $\lim\limits_{x\to 1^-} x^2+3x = 3^2+3(3) = 18$ ✗

$\lim\limits_{x\to 1^+} \frac{x^2-5x+6}{3-x} = \lim\limits_{x\to 3^+} \frac{\cancel{(x-3)}(x-2)}{-\cancel{(x-3)}} = \lim\limits_{x\to 3^+} \frac{x-2}{-1} = \frac{3-2}{-1} = -1$

$\Rightarrow$ Not Continuous at $x=3$ but it is left-continuous.

3.] Consider the two functions $f(x)$ and $g(x)$ whose graphs are given below:





a.) Let $h(x) = f(x) + g(x)$. Show, using appropriate limits, that $\lim\limits_{x\to 1} h(x)$ exists and calculate its value.

$\lim\limits_{x\to 1^-} h(x) = \lim\limits_{x\to 1^-} f(x)+g(x) = \lim\limits_{x\to 1^-} f(x) + \lim\limits_{x\to 1^-} g(x) = -3+2 = -1$

$\lim\limits_{x\to 1^+} h(x) = \lim\limits_{x\to 1^+} f(x)+g(x) = \lim\limits_{x\to 1^+} f(x) + \lim\limits_{x\to 1^+} g(x) = 0+(-1) = -1$

$\Rightarrow \boxed{\lim\limits_{x\to 1} h(x) = -1}$

b.) Is $h(x)$ continuous at $x=1$?

$h(1) = f(1)+g(1) = 0+2 = 2$

Since $\lim\limits_{x\to 1} h(x) = -1 \neq 2 = h(1)$, it follows that $h(x)$ is not continuous at $x=1$.

4.] Determine the removable discontinuities and redefine the function so that it is continuous at its removable discontinuities.

$$f(x) = \frac{x^2 - 5x}{x^3 - 3x^2 - 10x}$$

Factor: $f(x) = \frac{x(x-5)}{x(x^2-3x-10)} = \frac{x(x-5)}{x(x-5)(x+2)}$

Note: $f(-2) = f(0) = f(5) = \frac{0}{0} = \text{DNE}$

Domain: $(-\infty,-2)\cup(-2,0)\cup(0,5)\cup(5,\infty)$

- $\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{\cancel{x(x-5)}}{\cancel{x(x-5)}(x+2)} = \lim_{x\to 0} \frac{1}{x+2} = \frac{1}{2} \to$ Hole at $(0,\frac{1}{2})$ Removable Discontinuity
- $\lim_{x\to 5} f(x) = \lim_{x\to 5} \frac{\cancel{x(x-5)}}{\cancel{x(x-5)}(x+2)} = \lim_{x\to 5} \frac{1}{x+2} = \frac{1}{7} \to$ Hole at $(5,\frac{1}{7})$ Removable Discontinuity
- $\lim_{x\to -2} f(x) = \lim_{x\to -2} \frac{\cancel{x(x-5)}}{\cancel{x(x-5)}(x+2)} = \lim_{x\to -2} \frac{1}{x+2}$: $\lim_{x\to -2^-} \frac{1}{x+2} = -\infty$, $\lim_{x\to -2^+} \frac{1}{x+2} = \infty$ $\to$ Vertical Asymptote at $x=-2$

Define $f(0)=\frac{1}{2}$ and $f(5)=\frac{1}{7}$ to "fix" the removable discontinuities.

5.] Use the Intermediate Value Theorem to show that the following equation has a solution on the given interval:

$$\underbrace{2x^3 + x}_{f(x)} = 2, \qquad (-1, 1).$$

Let $f(x) = 2x^3 + x$. This function is continuous on $[-1,1]$ because it is a polynomial. Consider the following:

- $f(-1) = 2(-1)^3 + (-1) = -3$
- $f(1) = 2(1)^3 + (1) = 3$

By the IVT, we know that $f(x)$ takes on all values between -3 and 3 over the interval $(-1,1)$. Since 2 is between -3 and 3, we know this equation has a solution on $(-1,1)$.