§14.5: Three-Person Constant-Sum Game

1.] Consider the following three-person zero-sum game:

	R_1: C_1	R_1: C_2	R_2: C_1	R_2: C_2
L_1	$(1, 1, -2)$	$(-4, 3, 1)$	$(3, -2, -1)$	$(-6, -6, 12)$
L_2	$(2, -4, 2)$	$(-5, -5, 10)$	$(2, 2, -4)$	$(-2, 3, -1)$

– Nash Equilibria

a.) Determine the Nash equilibria.

$(L_2, C_1, R_1) \to (2, -4, 2)$ $\quad$ $(L_1, C_1, R_2) \to (3, -2, -1)$

b.) Assuming coalitions are possible, determine the three "two-person" zero-sum games based on each coalition.

C-R Coalition

Payout for L

		y_{11} C_1R_1	y_{21} C_2R_1	y_{12} C_1R_2	y_{22} C_2R_2	
x_1	L_1	1	−4	3	−6	−6
$1-x_1$	L_2	2	−5	2	−2	$\boxed{-5}$
		2	$\boxed{-4}$	3	−2	

C_1R_1: $x_1 + 2(1-x_1) \longrightarrow v \le -x_1 + 2$

C_2R_1: $-4x_1 - 5(1-x_1) \longrightarrow v \le x_1 - 5$

C_1R_2: $3x_1 + 2(1-x_1) \longrightarrow v \le x_1 + 2$

C_2R_2: $-6x_1 - 2(1-x_1) \longrightarrow v \le -4x_1 - 2$

$\boxed{-5 \le v \le -4} \longrightarrow$ Mixed Strategies needed

L-R Coalition

Payout for C

	L_1R_1	L_2R_1	L_1R_2	L_2R_2	
C_1	1	−4	−2	2	$\boxed{-4}$
C_2	3	−5	−6	3	−6
	3	$\boxed{-4}$	−2	3	

Pure Strategy: $(L_2, C_1, R_1) \longrightarrow \underbrace{\boxed{(2, -4, 2)}}^{\text{Payouts}}$

$\boxed{v = -4} \longrightarrow$ Pure Saddle Solution

L-C Coalition

Payout for R

		y_{11} L_1C_1	y_{21} L_2C_1	y_{12} L_1C_2	y_{22} L_2C_2	
x_1	R_1	−2	2	1	10	$\boxed{-2}$
$1-x_1$	R_2	−1	−4	12	−1	−4
		$\boxed{-1}$	2	12	10	

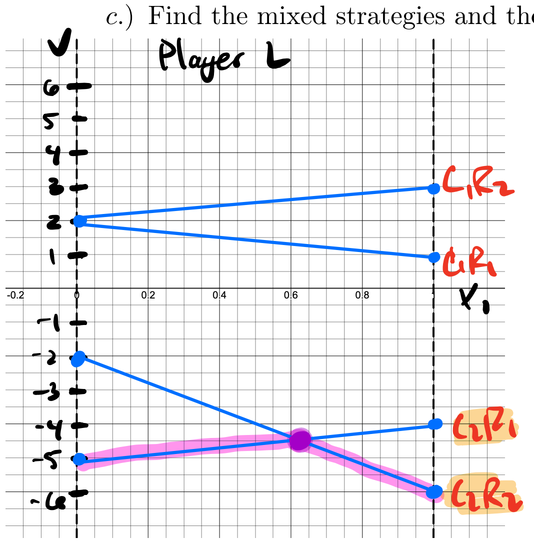
L_1C_1: $-2x_1 - (1-x_1) \longrightarrow v \le -x_1 - 1$

L_2C_1: $2x_1 - 4(1-x_1) \longrightarrow v \le 6x_1 - 4$

L_1C_2: $x_1 + 12(1-x_1) \longrightarrow v \le -11x_1 + 12$

L_2C_2: $10x_1 - (1-x_1) \longrightarrow v \le 11x_1 - 1$

$\boxed{-2 \le v \le -1} \longrightarrow$ Mixed Strategies needed *c.*) Find the mixed strategies and the value of the game for each player in the *C*-*R* coalition game.



Player L

C_1R_1: $V \le -x_1 + 2$

C_2R_1: $V \le x_1 - 5$

C_1R_2: $V \le x_1 + 2$

C_2R_2: $V \le -4x_1 - 2$

$x_1 - 5 = -4x_1 - 2$

$\Rightarrow \quad 5x_1 = 3$

$\Rightarrow \begin{cases} x_1 = 3/5 \\ x_2 = 2/5 \end{cases}$

Coalition C-R

$y_{11} = y_{12} = 0, \quad y_{22} = 1 - y_{21}$

L_1: $-4y_{21} - 6(1 - y_{21}) = 2y_{21} - 6$

L_2: $-5y_{21} - 2(1 - y_{21}) = -3y_{21} - 2$

$2y_{21} - 6 = -3y_{21} - 2$

$\Rightarrow \quad 5y_{21} = 4$

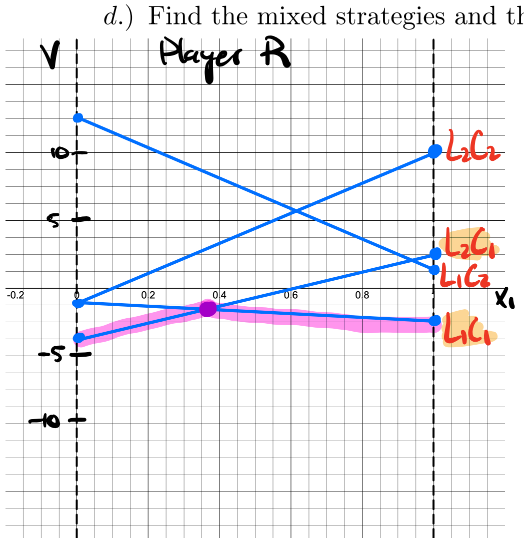
$\Rightarrow \begin{cases} y_{21} = 4/5 \\ y_{22} = 1/5 \end{cases}$

Strategies used: $L_1C_2R_1$, $L_1C_2R_2$, $L_2C_2R_1$, $L_2C_2R_2$

Payouts

Values: $\frac{3}{5}\cdot\frac{4}{5}(-4,3,1) + \frac{3}{5}\cdot\frac{1}{5}(-6,-6,12) + \frac{2}{5}\cdot\frac{4}{5}(-5,-5,10) + \frac{2}{5}\cdot\frac{1}{5}(-2,3,-1) = \boxed{(-4.4, -.64, 5.04)}$

d.) Find the mixed strategies and the value of the game for each player in the *L*-*C* coalition game.



Player R

L_1C_1: $V \le -x_1 - 1$

L_2C_1: $V \le 6x_1 - 4$

L_1C_2: $V \le -11x_1 + 12$

L_2C_2: $V \le 11x_1 - 1$

$-x_1 - 1 = 6x_1 - 4$

$\Rightarrow \quad 7x_1 = 3$

$\Rightarrow \begin{cases} x_1 = 3/7 \\ x_2 = 4/7 \end{cases}$

Coalition L-C

$y_{12} = y_{22} = 0, \quad y_{21} = 1 - y_{11}$

L_1: $-2y_{11} + 2(1 - y_{11}) = -4y_{11} + 2$

L_2: $-y_{11} - 4(1 - y_{11}) = 3y_{11} - 4$

$-4y_{11} + 2 = 3y_{11} - 4$

$\Rightarrow \quad 7y_{11} = 6$

$\Rightarrow \begin{cases} y_{11} = 6/7 \\ y_{21} = 1/7 \end{cases}$

Strategies used: $L_1C_1R_1$, $L_2C_1R_1$, $L_1C_1R_2$, $L_2C_1R_2$

Payouts

Values: $\frac{3}{7}\cdot\frac{6}{7}(1,1,-2) + \frac{3}{7}\cdot\frac{1}{7}(2,-4,2) + \frac{4}{7}\cdot\frac{6}{7}(3,-2,-1) + \frac{4}{7}\cdot\frac{1}{7}(2,2,-4) = \boxed{(2.12, -.69, -1.43)}$

e.) Determine the characteristic function of the game.

Payouts:

L vs. C-R: (-4.4, -.64, 5.04)

C vs. L-R: (2, -4, 2)

R vs. L-C: (2.12, -.69, -1.43)

Summary:

- L prefers a coalition with C
- C prefers a coalition with R
- R prefers a coalition with C
- The C-R coalition will likely form unless L can offer more winnings to L than R can.

Characteristic Function:

$v(\emptyset) = 0$	$v(C\text{-}R) = 4.4$
$v(L) = -4.4$	$v(L\text{-}R) = 4$
$v(C) = -4$	$v(L\text{-}C) = 1.43$
$v(R) = -1.43$	$v(L\text{-}C\text{-}R) = 0$