§11.3: LP Formulation of Two-Person Zero-Sum Games

1.] Consider the following two-person zero-sum game: Minimax = 10

	B_1	B_2	B_3	
	10	50	50	
A_1	5	50	50	5
A_2	1	1	.1	.1
A_3	10	1	10	1

maximin = 5

Verify that the strategies $(x_1, x_2, x_3) = (1/6, 0, 5/6)$ for A and $(y_1, y_2, y_3) = (49/54, 5/54, 0)$ for B are optimal, and determine the value of the game.

B's Strategy — A's Expectation

B_1: $5x_1 + x_2 + 10x_3 = 5(\frac{1}{6}) + 10(\frac{5}{6}) = \frac{55}{6}$

B_2: $50x_1 + x_2 + x_3 = 50(\frac{1}{6}) + \frac{5}{6} = \frac{55}{6}$

B_3: $50x_1 + .1x_2 + 10x_3 = 50(\frac{1}{6}) + 10(\frac{5}{6}) = \frac{100}{6}$

$\boxed{\text{Min} = \frac{55}{6}}$

A's Strategy — B's Expectation

A_1: $5y_1 + 50y_2 + 50y_3 = 5(\frac{49}{54}) + 50(\frac{5}{54}) = \frac{55}{6}$

A_2: $y_1 + y_2 + .1y_3 = \frac{49}{54} + \frac{5}{54} = 1$

A_3: $10y_1 + y_2 + 10y_3 = 10(\frac{49}{54}) + \frac{5}{54} = \frac{55}{6}$

$\boxed{\text{Max} = \frac{55}{6}}$

Value of the game: $\boxed{v = 55/6}$

2.] Colonel Blotto's army is fighting for the control of two strategic locations. Blotto has two regiments and the enemy has three. A location will fall to the army with more regiments. Otherwise, the result of the battle is a draw. Formulate the problem as a two-person zero-sum game, and solve it in Excel. Which army will win the battle?

Let (n_1, n_2) denote the allocation of regiments to each location. Then

Enemy

Blotto	B_1 (3,0)	B_2 (2,1)	B_3 (1,2)	B_4 (0,3)
A_1: (2,0)	-1	-1	0	0
A_2: (1,1)	0	-1	-1	0
A_3: (0,2)	0	0	-1	-1

Let x_i = probability that Blotto uses strategy A_i

Blotto's LP:

Max $z = v$

S.T.

$v \le -x_1 + 0x_2 + 0x_3$

$v \le -x_1 - x_2 + 0x_3$

$v \le 0x_1 - x_2 - x_3$

$v \le 0x_1 + 0x_2 - x_3$

$x_1 + x_2 + x_3 = 1$

$x_i \ge 0$, v urs.

Excel Solution:

$x_1 = 0.5$

$x_2 = 0$

$x_3 = 0.5$

$v = -0.5$

↳ Blotto loses.

3.] In the two-player, two-finger Morra game, each player shows one or two fingers, and simultaneously guesses the number of fingers the opponent will show. The player making the correct guess wins an amount equal to the total number of fingers shown. Otherwise, the game is a draw. Set up the problem as a two-person zero-sum game. Formulate the problem for player A and player B, and verify that the B formulation is the dual problem to the A formulation. Solve them both in Excel.

- Let (n_1, n_2) be each players strategy where n_1 is the number of fingers played and n_2 is the number of fingers guessed.
- Then the reward matrix for player A is below:

Player B

Player A		B_1 (1,1)	B_2 (1,2)	B_3 (2,1)	B_4 (2,2)
A_1	(1,1)	0	2	-3	0
A_2	(1,2)	-2	0	0	3
A_3	(2,1)	3	0	0	-4
A_4	(2,2)	0	-3	4	0

Decision Variables:

Let x_i = probability that Player A uses strategy A_i

Let y_j = probability that Player B uses strategy B_j

Player A's LP:

Maximize $Z = V$

Subject to

$$
\begin{aligned}
V \quad + 2x_2 - 3x_3 \quad &\leq 0 \\
V - 2x_1 \quad + 3x_4 &\geq 0 \\
V + 3x_1 \quad - 4x_4 &\geq 0 \\
V \quad - 3x_2 + 4x_3 \quad &\geq 0 \\
x_1 + x_2 + x_3 + x_4 &= 1 \\
x_1, x_2, x_3, x_4 \geq 0,\ V &\text{ urs}
\end{aligned}
$$

Player 's LP:

Minimize $Z = W$

Subject to

$$
\begin{aligned}
W \quad - 2y_2 + 3y_3 \quad &\geq 0 \\
W + 2y_1 \quad - 3y_4 &\geq 0 \\
W - 3y_1 \quad + 4y_4 &\geq 0 \\
W \quad + 3y_2 - 4y_3 \quad &\geq 0 \\
y_1 + y_2 + y_3 + y_4 &= 1 \\
y_1, y_2, y_3, y_4 \geq 0,\ W &\text{ urs}
\end{aligned}
$$

Excel Solution:

$x_2 = .6,\ x_3 = .4,\ x_1 = x_4 = 0;\ V = 0$

Excel Solution:

$y_2 = .6,\ y_3 = .4,\ y_1 = y_4 = 0;\ W = 0$