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Instructions: All solutions should be prepared carefully and neatly. All solution sets shall be completed on this packet and submitted by uploading a scan or picture of your written work to D2L by 11:59 PM on the due date below. **Submit only a single pdf file of your entire packet. Desmos graphs can be submitted separately.** The mobile app called *Genius Scan* works well. Use a PENCIL and if you make a mistake, use an eraser. This assignment is graded on effort, completeness, and neatness for a total of 5 points. Careless presentation (e.g. bad handwriting, pen scribbles, doodles, wasted space, etc) will result in a deduction of points at my discretion. Submitted work that does not demonstrate clearly the process by which one arrived at the answer may result in a loss of points. Any parts to any questions that are not answered will also result in a loss of points. Academic dishonesty will not be tolerated.

Problem Set VI

MAT 181 – Calculus I

Due: Friday, April 5 by 11:59 PM on D2L

Read: Sections 4.1–4.4

Solutions!

1. (This question spans two pages.) Find all the critical points of the following functions on the given intervals:

(a) $f(x) = 3x^4 + 8x^3 - 18x^2$ on $(-\infty, \infty)$

$f'(x) = 12x^3 + 24x^2 - 36x$

$\Rightarrow f'(x) = 12x(x^2 + 2x - 3)$

$\Rightarrow f'(x) = 12x(x+3)(x-1)$

$f'(x) = \text{DNE}$

N/A

$f'(x) = 0$

$12x(x+3)(x-1) = 0$

$x = 0,\ x = -3,\ x = 1$

Critical Points: $\boxed{c_1 = -3,\ c_2 = 0,\ c_3 = 1}$

(b) $f(x) = \frac{4}{3}x^3 + 5x^2 - 6x$ on the interval $[-1, 3]$

$f'(x) = 4x^2 + 10x - 6$

$f'(x) = \text{DNE}$

N/A

$f'(x) = 0$

$4x^2 + 10x - 6 = 0$

$2(2x^2 + 5x - 3) = 0$

$2(2x - 1)(x + 3) = 0$

$x = \frac{1}{2},\ \cancel{x = -3}$ outside interval

Critical Pts: $\boxed{c = \frac{1}{2}}$

(c) $g(x) = x^{1/3}(x - 2)^2$ on $(-\infty, \infty)$

$g'(x) = \frac{1}{3}x^{-2/3}(x-2)^2 + x^{1/3}\,2\cdot(x-2)\cdot 1$

$\Rightarrow g'(x) = \dfrac{(x-2)^2}{3x^{2/3}} + 2x^{1/3}(x-2)$

$\Rightarrow g'(x) = \dfrac{(x-2)^2 + 2x^{1/3}(x-2)\cdot 3x^{2/3}}{3x^{2/3}}$

$\Rightarrow g'(x) = \dfrac{(x-2)^2 + 6x(x-2)}{3x^{2/3}}$

$g'(x) = \text{DNE}$

$3x^{2/3} = 0$

$x = 0$

$g'(x) = 0$

$(x-2)^2 + 6x(x-2) = 0$

$\Rightarrow (x-2)(x-2+6x) = 0$

$\Rightarrow (x-2)(7x-2) = 0$

$x = 2 \quad x = \frac{2}{7}$

Critical Points: $\boxed{c_1 = 0,\ c_2 = \frac{2}{7},\ c_3 = 2}$

2. Consider the two functions, $f(x) = x^2 - x$ and $g(x) = x^{1/3}$.

(a) Does f satisfy the hypotheses of Rolle's Theorem on the interval $[-1, 1]$? If yes, explain briefly why. If not, explicitly state which part(s) of the hypothesis is violated. If it does satisfy the hypothesis, determine the value c that is guaranteed to exist.

Hypothesis:

1.) $f(x) = x^2 - x$ is continuous on $[-1,1]$ since it is a polynomial

2.) $f'(x) = 2x-1$ is continuous on $(-1,1)$ since it is a polynomial. Thus, $f(x)$ is differentiable on $(-1,1)$.

3.) $f(1) = 1^2 - 1 = 1-1 = 0$
$f(-1) = (-1)^2 - (-1) = 1+1 = 2$
not equal

$\boxed{\text{Thus, } f(x) = x^2 - x \text{ does not satisfy the hypothesis of Rolle's Theorem.}}$

(b) Does f satisfy the hypotheses of the Mean Value Theorem on the interval $[-1, 1]$? If yes, explain briefly why. If no, explicitly state which part(s) of the hypothesis is violated. If it does satisfy the hypothesis, determine the value c that is guaranteed to exist.

Hypothesis:

1.) $f(x) = x^2 - x$ is continuous on $[-1,1]$ since it is a polynomial

2.) $f'(x) = 2x-1$ is continuous on $(-1,1)$ since it is a polynomial. Thus, $f(x)$ is differentiable on $(-1,1)$.

Conclusion: $f'(c) = \frac{f(b)-f(a)}{b-a}$

$\Rightarrow 2c - 1 = \frac{f(1)-f(-1)}{1-(-1)}$

$\Rightarrow 2c-1 = \frac{0-2}{2}$

$\Rightarrow 2c-1 = -1$

$\Rightarrow 2c = 0$

$\Rightarrow \boxed{c=0}$

(c) Does g satisfy the hypotheses of the Mean Value Theorem on the interval $[-1, 1]$? If yes, explain briefly why. If no, explicitly state which part(s) of the hypothesis is violated. If it does satisfy the hypothesis, determine the value c that is guaranteed to exist.

Hypothesis:

1.) $f(x) = x^{1/3}$ is continuous on $[-1,1]$

2.) $f(x) = x^{1/3}$ is not differential because

$$f'(x) = \frac{1}{3}x^{-2/3} = \frac{1}{3x^{2/3}}$$

which DNE at $x=0$

3.) $f(-1) = (-1)^{1/3} = -1$
$f(1) = (1)^{1/3} = 1$
$\neq$

$\boxed{\text{Thus, } f(x) = x^{1/3} \text{ does not satisfy the hypothesis of Rolle's Theorem.}}$

3. (This question spans two pages.) Parts a and b below ask you to use either the first derivative test or the second derivative test to identify local extrema.

   (a) Find all the critical points and candidate inflection points of the function

$$f(x) = \frac{2}{3}x^3 + 4x^2 - 10x + \frac{5}{3}.$$

Use the provided number lines below to determine on the intervals of monotonicity and concavity. Using the **First Derivative Test**, determine all local extrema. Finally, determine where the function has any inflection points. Submit a desmos graph that supports your answer.

Montonicity:

$f'(x) = 2x^2 + 8x - 10 = 0$

$2(x^2 + 4x - 5) = 0$

$2(x+5)(x-1) = 0$

$\boxed{c_1 = -5,\ c_2 = 1}$

critical pts

Concavity

$f''(x) = 4x + 8 = 0$

$4x = -8$

$x = -2$

$\boxed{i_1 = -2}$

Inflection Pt

Summary

- Increasing on $(-\infty,-5)\cup(1,\infty)$
- Decreasing on $(-5,1)$
- Concave up on $(-2,\infty)$
- Concave down on $(-\infty,-2)$

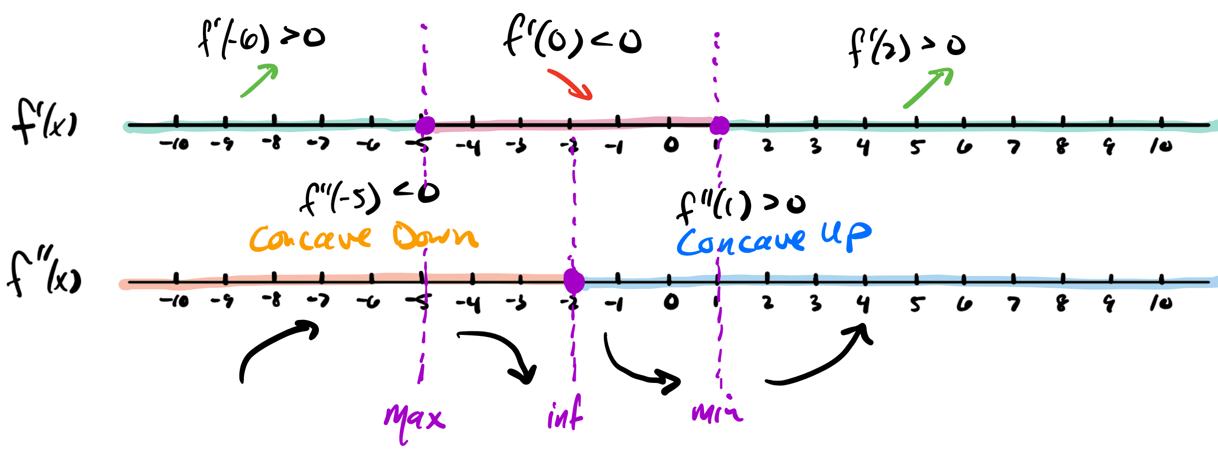
First Derivative Test

$f'(-6) = (+)(-)(-) > 0$

$f'(0) = (+)(+)(-) < 0$

$f'(2) = (+)(+)(+) > 0$

$\boxed{\begin{array}{l}\text{local max at } x=-5\\ \text{local min at } x=1\end{array}}$



(b) Find all the critical points and candidate inflection points of the function

$$f(x) = \frac{3}{4}x^4 - 4x^3 - \frac{21}{2}x^2 + 30x + 4.$$

Use the provided number lines below to determine on the intervals of monotonicity and concavity. Using the **Second Derivative Test**, determine all local extrema. Finally, determine where the function has any inflection points. Submit a desmos graph that supports your answer.

Montonicity:

$$f'(x) = 3x^3 - 12x^2 - 21x + 30$$
$$= 3(x^3 - 4x^2 - 7x + 10) = 0$$
$$= 3(x-1)(x^2 - 3x - 10) = 0$$
$$= 3(x-1)(x-5)(x+2) = 0$$

$$\boxed{c_1 = -2,\ c_2 = 1,\ c_3 = 5}$$

Critical Pts

$$f'(-3) = (+)(-)(-)(-) < 0$$
$$f'(0) = (+)(-)(-)(+) > 0$$
$$f'(2) = (+)(+)(-)(+) < 0$$
$$f'(6) = (+)(+)(+)(+) > 0$$

Concavity

$$f''(x) = 9x^2 - 24x - 21$$
$$= 3(3x^2 - 8x - 7) = 0$$

$$x = \frac{-(-8)}{2(3)} \pm \frac{\sqrt{(-8)^2 - 4(-7)(3)}}{2(3)}$$
$$x = \frac{4}{3} \pm \frac{\sqrt{64+84}}{6}$$
$$x = \frac{4}{3} \pm \frac{\sqrt{148}}{6}$$
$$x = \frac{4}{3} \pm \frac{\sqrt{37}}{3}$$

$$\boxed{i_1 = \frac{4-\sqrt{37}}{3} \approx -.69 \qquad i_2 = \frac{4+\sqrt{37}}{3} \approx 3.36}$$

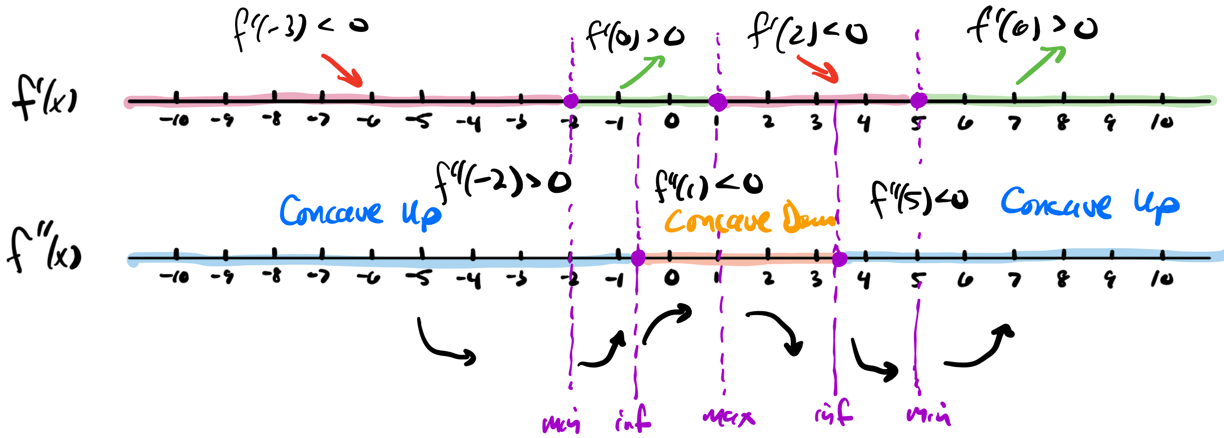
Inflection Pts

Summary

- Increasing on $(-2,1) \cup (5,\infty)$
- Decreasing on $(-\infty,-2) \cup (1,5)$
- Concave up on $\left(-\infty, \frac{4-\sqrt{37}}{3}\right) \cup \left(\frac{4+\sqrt{37}}{3}, \infty\right)$
- Concave down on $\left(\frac{4-\sqrt{37}}{3}, \frac{4+\sqrt{37}}{3}\right)$

Second Derivative Test:

$$f''(-2) = 3(3(-2)^2 - 8(-2) - 7) = 63 > 0 \longrightarrow \text{local min at } x = -2$$
$$f''(1) = 3(3(1)^2 - 8(1) - 7) = -36 < 0 \longrightarrow \text{local max at } x = 1$$
$$f''(5) = 3(3(5)^2 - 8(5) - 7) = 84 > 0 \longrightarrow \text{local min at } x = 5$$



4. Evaluate the following limits using techniques from Chapter 2 or using L'Hôpital's Rule. If you must use L'Hôpital's Rule, indicate the indeterminate form.

(a) $\lim_{x\to 0} \frac{\sin(x)}{x}$ $\left[= \frac{\sin(0)}{0} = \frac{0}{0}\right]$

L'H $\lim_{x\to 0} \frac{\cos(x)}{1} = \frac{\cos(0)}{1} = \frac{1}{1} = \boxed{1}$

(b) $\lim_{x\to 0} \frac{5^x - 1}{x}$ $\left[= \frac{5^0-1}{0} = \frac{1-1}{0} = \frac{0}{0}\right]$

L'H $\lim_{x\to 0} \frac{\ln(5)\cdot 5^x - 0}{1} = \lim_{x\to 0} \ln(5)\cdot 5^x = \ln(5)\cdot 5^0 = \boxed{\ln(5)}$

(c) $\lim_{x\to 0} \frac{\cos(x) - 1}{x}$ $\left[= \frac{\cos(0)-1}{0} = \frac{1-1}{0} = \frac{0}{0}\right]$

L'H $\lim_{x\to 0} \frac{-\sin(x) - 0}{1} = \lim_{x\to 0} -\sin(x) = -\sin(0) = -0 = \boxed{0}$

(d) $\lim_{x\to 36} \frac{\sqrt{x} - 6}{x - 36}$ $\left[= \frac{\sqrt{36}-6}{36-36} = \frac{6-6}{0} = \frac{0}{0}\right]$

L'H $= \lim_{x\to 36} \frac{\frac{1}{2\sqrt{x}} - 0}{1 - 0} = \lim_{x\to 36} \frac{1}{2\sqrt{x}} = \frac{1}{2\sqrt{36}} = \frac{1}{2\cdot 6} = \boxed{\frac{1}{12}}$

(e) $\lim_{x\to 3} \frac{2^x - 8}{12 - 4x}$ $\left[= \frac{2^3-8}{3-3} = \frac{8-8}{0} = \frac{0}{0}\right]$

L'H $\lim_{x\to 3} \frac{\ln(2)\cdot 2^x - 0}{0 - 4} = \lim_{x\to 3} \frac{\ln(2)\cdot 2^x}{-4} = \lim_{x\to 3} \frac{\ln(2)\cdot 2^3}{-4} = \boxed{-2\ln(2)}$

(f) $\lim_{x\to \infty} \frac{5^x}{6^x} = \lim_{x\to\infty} \left(\frac{5}{6}\right)^x$

$= \boxed{0}$ base is less than 1.

(g) $\lim_{x\to 0^+} \frac{1-\cos(x)}{\csc(x)}$

$= \lim_{x\to 0^+} \sin(x)(1-\cos(x)) = \sin(0)(1-\cos(0)) = 0\cdot(1-1) = 0\cdot 0 = \boxed{0}$

(h) $\lim_{x\to\infty} 3x^4e^{-2x}$

$= \lim_{x\to\infty} \frac{3x^3}{e^{2x}}$ $\left[= \frac{\infty}{\infty}\right]$

L'H

$= \lim_{x\to\infty} \frac{9x^2}{2e^{2x}}$ $\left[= \frac{\infty}{\infty}\right]$

L'H

$= \lim_{x\to\infty} \frac{18x}{4e^{2x}}$

$= \lim_{x\to\infty} \frac{18x}{4e^{2x}}$ $\left[= \frac{\infty}{\infty}\right]$

L'H

$= \lim_{x\to\infty} \frac{18}{8e^{2x}}$

$= \boxed{0}$

(i) $\lim_{x\to\infty} x^{\frac{1}{2x}}$ $[= \infty^0]$

$= \lim_{x\to\infty} e^{\ln(x^{1/2x})}$

$= e^{\lim_{x\to\infty} \frac{\ln(x)}{2x}}$

$\lim_{x\to\infty} \frac{\ln(x)}{2x}$ $\left[= \frac{\infty}{\infty}\right]$

$= \lim_{x\to\infty} \frac{\frac{1}{x}}{2}$

$= \lim_{x\to\infty} \frac{1}{2x}$

$= 0$

$= e^0 = \boxed{1}$

(j) $\lim_{x\to\infty} \left(1+\frac{2}{x}\right)^x$ $[= (1+0)^\infty = 1^\infty]$

$= \lim_{x\to\infty} e^{\ln(1+\frac{2}{x})^x}$

$= e^{\lim_{x\to\infty} x\ln(1+\frac{2}{x})}$

$\lim_{x\to\infty} x\ln(1+\frac{2}{x})$ $[= \infty\cdot 0]$

$= \lim_{x\to\infty} \frac{\ln(1+\frac{2}{x})}{\frac{1}{x}}$ $\left[= \frac{0}{0}\right]$

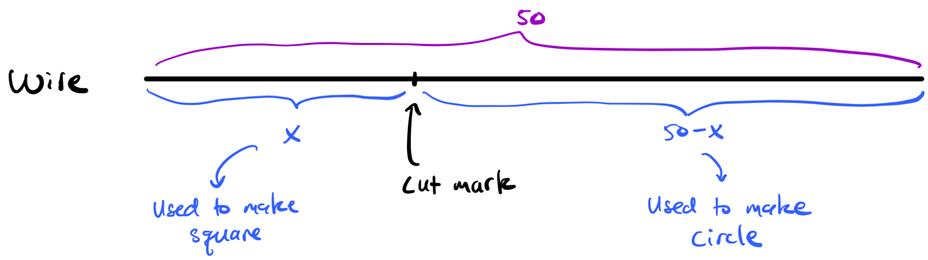
L'H

$= \lim_{x\to\infty} \frac{\frac{1}{1+\frac{2}{x}}\cdot\left(-\frac{2}{x^2}\right)}{-\frac{1}{x^2}}$

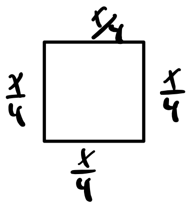
$= \lim_{x\to\infty} \frac{2}{1+\frac{2}{x}} = \frac{2}{1+0} = 2$

$= \boxed{e^2}$

5. Application Problem: A 50-inch piece of wire is to be cut into two pieces (a portion of length x and a portion of length $50 - x$, see figure below), which are then bent into a square and a circle, respectively. Where should the wire be cut (i.e. what is x?) in order to minimize the sum of the areas of these two shapes? Where should it be cut to maximize the sum of the areas? To solve both problems, formulate a function $f(x)$ that represents the sum of the two areas in terms of x, then use the Closed Interval Method to determine the absolute minimum (and maximum) on the interval $x \in [0, 50]$.

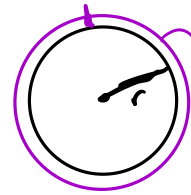


Square:



$$A_s = \left(\frac{x}{4}\right)^2 = \frac{x^2}{16}$$

Circle:



$$C = 50 - x$$
$$2\pi r = 50 - x$$
$$r = \frac{50 - x}{2\pi}$$

$$A_c = \pi r^2 = \pi\left(\frac{50-x}{2\pi}\right)^2 = \frac{\pi(50-x)^2}{4\pi^2} = \frac{(50-x)^2}{4\pi}$$

$$f(x) = A_s + A_c$$

$$\Rightarrow f(x) = \frac{x^2}{16} + \frac{(50-x)^2}{4\pi}$$

$$\Rightarrow f'(x) = \frac{2x}{16} + \frac{2(50-x)}{4\pi}(-1)$$

$$\Rightarrow f'(x) = \frac{1}{8}x - \frac{1}{2\pi}(50-x) = 0$$

$$\frac{1}{8}x - \frac{50}{2\pi} + \frac{1}{2\pi}x = 0$$

$$\left(\frac{1}{8} + \frac{1}{2\pi}\right)x = \frac{25}{\pi}$$

$$\frac{2\pi + 8}{16\pi}x = \frac{25}{\pi}$$

$$x = \frac{25}{\pi}\left(\frac{16\pi}{2\pi+8}\right)$$

$$x = \frac{200}{\pi+4} \approx \boxed{28}$$ critical pt

Closed Interval Method:

$$f(0) = \frac{50^2}{4\pi} = 198.94 \quad \text{max}$$

$$f(28) = \frac{28^2}{16} + \frac{(22)^2}{4\pi} = 87.52 \quad \text{min}$$

$$f(50) = \frac{50^2}{16} = 156.25$$

The absolute minimum occurs at $x = 28$ inches with total area of 87.52 in^2

The absolute maximum occurs at $x = 0$ inches with total area of 198.94 in^2 (only make a circle!)