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Instructions: All solutions should be prepared carefully and neatly. All solution sets shall be completed on this packet and submitted by uploading a scan or picture of your written work to D2L by 11:59 PM on the due date below. **Submit only a single pdf file of your entire packet. Desmos graphs can be submitted separately.** The mobile app called *Genius Scan* works well. Use a PENCIL and if you make a mistake, use an eraser. This assignment is graded on effort, completeness, and neatness for a total of 5 points. Careless presentation (e.g. bad handwriting, pen scribbles, doodles, wasted space, etc) will result in a deduction of points at my discretion. Submitted work that does not demonstrate clearly the process by which one arrived at the answer may result in a loss of points. Any parts to any questions that are not answered will also result in a loss of points. Academic dishonesty will not be tolerated.

Problem Set IV

MAT 181-050 – Calculus I

Due: Monday, March 4 by 11:59 PM on D2L

Read: Sections 3.2, 3.3, and 3.4

Solutions!

1. Using the appropriate rules of differentiation, find the derivative of the following functions: [(7)]

   (a) $f(x) = 1$

   $f'(x) = 0$

   (b) $u(x) = \pi^3 + 12$

   $u'(x) = 0$

   (c) $g(t) = t^{42070}$

   $g'(t) = 42070\, t^{42069}$

   (d) $p(t) = 42070^t$

   $p'(t) = \ln(42070) \cdot 42070^t$

   (e) $y(w) = 4w^6 + 14\sqrt{w} - 2^w$, where $w \geq 0$.

   $y'(w) = 24w^5 + \frac{7}{\sqrt{w}} - \ln(2) \cdot 2^w$

   (f) $q(s) = \dfrac{s^5 - 2s^2}{s^2}$, where $s \neq 0$.

   $q(s) = s^3 - 2 \longrightarrow q'(s) = 3s^2$, where $s \neq 0$.

   (g) $F(x) = \dfrac{e^x}{3} - 11x^3$

   $F'(x) = \frac{e^x}{3} - 33x^2$

   (h) $h(t) = (t - t^3)(t^2 - 1)$

   $h(t) = t^3 - t - t^5 + t^3$

   $h(t) = -t^5 + 2t^3 - t \longrightarrow h'(t) = -5t^4 + 6t^2 - 1$

2. (This problem spans two pages.) Differentiate the following functions and simplify the expression as much as possible: [ ]

(a) $f(x) = 2^x\sqrt{x}$

$$f'(x) = (2^x)'\sqrt{x} + (\sqrt{x})' \cdot 2^x$$

$$f'(x) = \ln(2)\cdot 2^x \sqrt{x} + \frac{1}{2\sqrt{x}} \cdot 2^x$$

$$\Rightarrow f'(x) = 2^x\left(\ln(2)\cdot\sqrt{x} + \frac{1}{2\sqrt{x}}\right)$$

$$f'(x) = 2^x\left(\frac{2\ln(2)x+1}{2\sqrt{x}}\right)$$

$$\Rightarrow \boxed{f'(x) = \frac{2^{x-1}(2\ln(2)\cdot x+1)}{\sqrt{x}}}$$

(b) $g(x) = \dfrac{100x}{x-1}$

$$g'(x) = \frac{(x-1)(100x)' - (x-1)'(100x)}{(x-1)^2}$$

$$\Rightarrow g'(x) = \frac{(x-1)(100) - 100x}{(x-1)^2}$$

$$g'(x) = \frac{\cancel{100x} - 100 - \cancel{100x}}{(x-1)^2}$$

$$\boxed{g'(x) = \frac{-100}{(x-1)^2}}$$

(c) $h(x) = \dfrac{\sqrt{x}+x}{\sqrt{x}-x}$

$$h'(x) = \frac{(\sqrt{x}-x)(\sqrt{x}+x)' - (\sqrt{x}+x)(\sqrt{x}-x)'}{(\sqrt{x}-x)^2}$$

$$\Rightarrow h'(x) = \frac{(\sqrt{x}-x)\left(\frac{1}{2\sqrt{x}}+1\right) - (\sqrt{x}+x)\left(\frac{1}{2\sqrt{x}}-1\right)}{(\sqrt{x}-x)^2}$$

$$\Rightarrow h'(x) = \frac{\left(\frac{1}{2}+\sqrt{x}-\frac{1}{2}\sqrt{x}-x\right) - \left(\frac{1}{2}-\sqrt{x}+\frac{1}{2}\sqrt{x}-x\right)}{(\sqrt{x}-x)^2}$$

$$h'(x) = \frac{\cancel{\frac{1}{2}}+\sqrt{x}-\frac{1}{2}\sqrt{x}-\cancel{x}-\cancel{\frac{1}{2}}+\sqrt{x}-\frac{1}{2}\sqrt{x}+\cancel{x}}{(\sqrt{x}-x)^2}$$

$$\Rightarrow \boxed{h'(x) = \frac{\sqrt{x}}{(\sqrt{x}-x)^2}}$$

(d) $k(x) = \dfrac{2x+1}{x^3e^x}$

$$k'(x) = \frac{(x^3e^x)(2x+1)' - (2x+1)(x^3e^x)'}{(x^3e^x)^2}$$

Product Rule!

$$\Rightarrow k'(x) = \frac{x^3e^x\cdot 2 - (2x+1)(3x^2e^x + x^3e^x)}{x^6e^{2x}}$$

$$\Rightarrow k'(x) = \frac{2x^3e^x - 6x^3e^x - 2x^4e^x - 3x^2e^x - x^3e^x}{x^6e^{2x}}$$

$$k'(x) = \frac{-2x^4e^x - 5x^3e^x - 3x^2e^x}{x^6e^{2x}}$$

$$k'(x) = \frac{-x^2e^x(2x^2+5x+3)}{x^6e^{2x}}$$

$$\boxed{k'(x) = \frac{-(2x^2+5x+3)}{x^4e^x}}$$

(e) $k(x) = \dfrac{1-\sin(x)}{1-\cos(x)}$

$$k'(x) = \frac{(1-\cos(x))(1-\sin(x))' - (1-\sin(x))(1-\cos(x))'}{(1-\cos(x))^2}$$

$$\Rightarrow k'(x) = \frac{(1-\cos(x))(-\cos(x)) - (1-\sin(x))(\sin(x))}{(1-\cos(x))^2}$$

$$k'(x) = \frac{-\cos(x) + \cos^2(x) - \sin(x) + \sin^2(x)}{(1-\cos(x))^2}$$

$$\boxed{k'(x) = \frac{1-\cos(x) - \sin(x)}{(1-\cos(x))^2}}$$

(f) $f(x) = e^{5x+2}\tan(\sqrt{x})$

$$f'(x) = (e^{5x+2})'(\tan(\sqrt{x})) + (\tan(\sqrt{x}))'(e^{5x+2})$$

$$\Rightarrow f'(x) = e^{5x+2}\cdot(5x+2)'\cdot(\tan(\sqrt{x})) + \sec^2(\sqrt{x})\cdot(\sqrt{x})'\cdot(e^{5x+2})$$

$$\Rightarrow f'(x) = 5e^{5x+2}\tan(\sqrt{x}) + \sec^2(\sqrt{x})\cdot\left(\tfrac{1}{2\sqrt{x}}\right)\cdot e^{5x+2}$$

$$\boxed{f'(x) = e^{5x+2}\left(5\tan(\sqrt{x}) + \frac{\sec^2(\sqrt{x})}{2\sqrt{x}}\right)}$$

(g) $g(x) = (2x^{40} + 13x^3 + \sqrt[3]{x} - 5)^{20}$

$$g'(x) = 20(2x^{40} + 13x^3 + \sqrt[3]{x} - 5)^{19}\cdot(2x^{40} + 13x^3 + \sqrt[3]{x} - 5)'$$

$$\Rightarrow \boxed{g'(x) = 20(2x^{40} + 13x^3 + \sqrt[3]{x} - 5)^{19}\cdot\left(80x^{39} + 39x^2 + \tfrac{1}{3}x^{-2/3}\right)}$$

(h) $h(x) = \cos(5\sin(x))$

$$h'(x) = -\sin(5\sin(x))\cdot(5\sin(x))'$$

$$\Rightarrow h'(x) = -\sin(5\sin(x))\cdot 5\cos(x)$$

$$\Rightarrow \boxed{h'(x) = -5\cos(x)\sin(5\sin(x))}$$

(i) $k(x) = \sqrt[5]{x^3 + \sin(2x^5 - 3)}$

$$k'(x) = \tfrac{1}{5}(x^3 + \sin(2x^5-3))^{-4/5}\cdot(x^3 + \sin(2x^5-3))'$$

$$\Rightarrow k'(x) = \tfrac{1}{5}(x^3 + \sin(2x^5-3))^{-4/5}\cdot(3x^2 + \cos(2x^5-3)\cdot(2x^5-3)')$$

$$\Rightarrow \boxed{k'(x) = \frac{3x^2 + 10x^4\cos(2x^5-3)}{5(x^3+\sin(2x^5-3))^{4/5}}}$$

3. In each problem below, find the equation of the tangent line to the curve at the given point. **For each problem, submit a Desmos graph that shows the function and the tangent line on the same plot.**

   (a) Consider $f(x) = \dfrac{2x^2}{3x-1}$ at the point $\big(1, f(1)\big)$.

Derivative Function:

- $f'(x) = \dfrac{(3x-1)(2x^2)' - (2x^2)(3x-1)'}{(3x-1)^2} = \dfrac{(3x-1)(4x)-(2x^2)(3)}{(3x-1)^2} = \dfrac{6x^2-4x}{(3x-1)^2} = \dfrac{2x(3x-2)}{(3x-1)^2}$

Equation of Tangent Line:

- $C = 1$
- $f(c) = f(1) = \dfrac{2(1)^2}{3(1)-1} = \dfrac{2}{3-1} = \dfrac{2}{2} = 1$
- $f'(c) = f'(1) = \dfrac{2(1)(3(1)-2)}{(3(1)-1)^2} = \dfrac{2(1)(1)}{2^2} = \dfrac{2}{4} = \dfrac{1}{2}$

$y - f(1) = f'(1)(x-1)$

$y - 1 = \frac{1}{2}(x-1)$

$\Rightarrow y - 1 = \frac{1}{2}x - \frac{1}{2}$

$\Rightarrow \boxed{y = \frac{1}{2}x + \frac{1}{2}}$

   (b) Consider $f(x) = 2\csc(x) - \sin(x)$ at the point $\left(\frac{\pi}{2}, f\left(\frac{\pi}{2}\right)\right)$.

Derivative Function:

- $f'(x) = -2\csc(x)\cot(x) - \cos(x)$

Equation of Tangent Line:

- $C = \pi/2$
- $f(c) = f(\pi/2) = 2\csc(\pi/2) - \sin(\pi/2) = 2\left(\frac{1}{\sin(\pi/2)}\right) - \sin(\pi/2) = 2\left(\frac{1}{1}\right) - 1 = 2 - 1 = 1$
- $f'(c) = f'(\pi/2) = -2\csc(\pi/2)\cot(\pi/2) - \cos(\pi/2) = -2\left(\frac{1}{\sin(\pi/2)}\right)\left(\frac{\cos(\pi/2)}{\sin(\pi/2)}\right) - \cos(\pi/2) = -2\left(\frac{1}{1}\right)\left(\frac{0}{1}\right) - 0 = 0$

$y - f(\pi/2) = f'(\pi/2)(x - \pi/2)$

$\Rightarrow y - 1 = 0(x - \pi/2)$

$\Rightarrow \boxed{y = 1}$

4. Consider the function $f(x) = \dfrac{x}{2x-1}$ and the point $(-7, 1)$. This point *does not* lie on the graph of $y = f(x)$. Suppose $(c, f(c))$ is a point on the graph of $y = f(x)$ such that the tangent line to f at c goes through the point $(-7, 1)$. Show that two possible values for c exist and find them.

The slope of the TL at $(c, f(c))$, which we know is $f'(c)$, must be equal to the slope of the line connecting the points $(c, f(c))$ and $(-7, 1)$, which we know is $\dfrac{1-f(c)}{-7-c}$. Therefore,

$$f'(x) = \frac{(2x-1)(x)' - (x)(2x-1)'}{(2x-1)^2} = \frac{(2x-1)(1) - x(2)}{(2x-1)^2} = \frac{-1}{(2x-1)^2}$$

Setting the slopes equal to each other at $(c, f(c))$ gives:

$$f'(c) = \frac{1-f(c)}{-7-c}$$

$$\Rightarrow \quad f'(c)(-7-c) = 1 - f(c)$$

$$\Rightarrow \quad \frac{-1}{(2c-1)^2}(-7-c) = 1 - \frac{c}{2c-1}$$

$$\Rightarrow \quad (2c-1)^2\left(\frac{-1}{(2c-1)^2}(-7-c)\right) = \left(1 - \frac{c}{2c-1}\right)(2c-1)^2$$

$$\Rightarrow \quad 7+c = (2c-1)^2 - c(2c-1)$$

$$\Rightarrow \quad 7+c = 4c^2 - 4c + 1 - 2c^2 + c$$

$$\Rightarrow \quad 2c^2 - 4c - 6 = 0$$

$$\Rightarrow \quad 2(c^2 - 2c - 3) = 0$$

$$\Rightarrow \quad 2(c-3)(c+1) = 0$$

$$\Rightarrow \quad c-3 = 0 \qquad c+1 = 0$$

$$\Rightarrow \quad \boxed{c=3} \qquad \boxed{c=-1}$$

5. Application Problem: The position function for damped harmonic motion of an object of mass m is

$$y(t) = Ae^{-\frac{k}{2m}t}\cos(\omega t)$$

where A is the amplitude, and k and ω are constants specific to the motion. Find the velocity and acceleration functions for this motion.

Position: $y(t) = Ae^{-\frac{k}{2m}t}\cos(\omega t)$

Velocity: $v(t) = y'(t)$

$$= \left(Ae^{-\frac{k}{2m}t}\cos(\omega t)\right)'$$

Product Rule

$$= \left(Ae^{-\frac{k}{2m}t}\right)'(\cos(\omega t)) + \left(Ae^{-\frac{2k}{m}t}\right)(\cos(\omega t))'$$

Chain Rule

$$= A\left(-\frac{k}{2m}\right)e^{-\frac{k}{2m}t}\cos(\omega t) + Ae^{-\frac{2k}{m}t}(-\sin(\omega t))\cdot\omega$$

$$= -\frac{Ak}{2m}e^{-\frac{k}{2m}t}\cos(\omega t) - A\omega e^{-\frac{2k}{m}t}\sin(\omega t)$$

$$\boxed{v(t) = -Ae^{-\frac{k}{2m}t}\left(\frac{k}{2m}\cos(\omega t) + \omega\sin(\omega t)\right)}$$

Acceleration: $a(t) = y''(t)$

$$= v'(t)$$

$$= \left(-Ae^{-\frac{k}{2m}t}\left(\frac{k}{2m}\cos(\omega t) + \omega\sin(\omega t)\right)\right)'$$

Product Rule

$$= \left(-Ae^{-\frac{k}{2m}t}\right)'\left(\frac{k}{2m}\cos(\omega t) + \omega\sin(\omega t)\right) + \left(-Ae^{-\frac{k}{2m}t}\right)\left(\frac{k}{2m}\cos(\omega t) + \omega\sin(\omega t)\right)'$$

$$= \frac{Ak}{2m}e^{-\frac{k}{2m}t}\left(\frac{k}{2m}\cos(\omega t) + \omega\sin(\omega t)\right) + \left(-Ae^{-\frac{k}{2m}t}\right)\left(-\frac{k\omega}{2m}\sin(\omega t) + \omega^2\cos(\omega t)\right)$$

$$= Ae^{-\frac{k}{2m}t}\left(\frac{k^2}{4m^2}\cos(\omega t) + \frac{k\omega}{2m}\sin(\omega t) + \frac{k\omega}{2m}\sin(\omega t) - \omega^2\cos(\omega t)\right)$$

$$\boxed{a(t) = Ae^{-\frac{2k}{m}t}\left[\left(\frac{k^2}{4m^2} - \omega^2\right)\cos(\omega t) + \frac{k\omega}{m}\sin(\omega t)\right]}$$