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Instructions: All solutions should be prepared carefully and neatly. All solution sets shall be completed on this packet and submitted by uploading a scan or picture of your written work to D2L by 11:59 PM on the due date below. **Submit only a single pdf file of your entire packet. Desmos graphs can be submitted separately.** The mobile app called *Genius Scan* works well. Use a PENCIL and if you make a mistake, use an eraser. This assignment is graded on effort, completeness, and neatness for a total of 5 points. Careless presentation (e.g. bad handwriting, pen scribbles, doodles, wasted space, etc) will result in a deduction of points at my discretion. Submitted work that does not demonstrate clearly the process by which one arrived at the answer may result in a loss of points. Any parts to any questions that are not answered will also result in a loss of points. Academic dishonesty will not be tolerated.

Problem Set III

MAT 181 – Calculus I

Due: Friday, February 23 by 11:59 PM on D2L

Read: Sections 2.1, 2.5, 2.6, and 3.1

Solutions!

1. Consider the following piecewise function below. Determine if the function is continuous at $x = 0$ and $x = 4$ by computing, explicitly, all necessary limits. (Note: you are not allowed to compute limits graphically.) If it is not continuous at a point, determine if it is left- or right-continuous or neither. Show all work to receive full credit.

$$f(x) = \begin{cases} \dfrac{\sin(x)}{x} & \text{if } x < 0 \\ \dfrac{x^3 - x^2 - 12x}{x^2 - 3x - 4} & \text{if } 0 \le x < 4 \\ \sqrt{x} + \dfrac{18}{5} & \text{if } x \ge 4 \end{cases}$$

Near $x=0$:

- $f(0) = \dfrac{0^3 - 0^2 - 12(0)}{0^2 - 3(0) - 4} = \dfrac{0}{-4} = \boxed{0}$

- $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \dfrac{\sin(x)}{x} \approx \dfrac{\sin(-.00001)}{-.00001} \to \boxed{1}$

- $\lim_{x\to 0^+} f(x) = \dfrac{0^3 - 0^2 - 12(0)}{0^2 - 3(0) - 4} = \dfrac{0}{-4} = \boxed{0}$

The limit as $x \to 0$ doesn't exist, therefore the function is not continuous at $x=0$. However, it is right continuous at $x=0$.

Near $x=4$:

- $f(4) = \sqrt{4} + \dfrac{18}{5} = 2 + \dfrac{18}{5} = \boxed{\dfrac{28}{5}}$

- $\lim_{x\to 4^-} f(x) = \lim_{x\to 4^-} \dfrac{x^3 - x^2 - 12x}{x^2 - 3x - 4} = \lim_{x\to 4^-} \dfrac{x(x-4)(x+3)}{(x-4)(x+1)} = \lim_{x\to 4^-} \dfrac{x(x+3)}{x+1} = \dfrac{4(4+3)}{4+1} = \boxed{\dfrac{28}{5}}$

- $\lim_{x\to 4^+} f(x) = \sqrt{4} + \dfrac{18}{5} = \boxed{\dfrac{28}{5}}$

The function is continuous at $x=4$ since the limit is equivalent to the function value.

2. Suppose that a sailboat is observed, over a period of 5 minutes, to travel a distance from a starting point according to the function $s(t) = t^3 + 60t$, where t is time in minutes and s is the distance traveled in meters.

(a) What is the average velocity of the boat between 1 and 3 minutes?

$[\text{Avg Vel}] = \frac{\Delta v}{\Delta t} = \frac{s(3)-s(1)}{3-1} = \frac{207-61}{2} = \frac{146}{2} = 73 \text{ meters/min}$

$s(3) = 3^3 + 60(3) = 27 + 180 = 207$

$s(1) = 1^3 + 60(1) = 1 + 60 = 61$

(b) What is the average velocity of the boat between 3 and 5 minutes?

$[\text{Avg Vel}] = \frac{\Delta v}{\Delta t} = \frac{s(5)-s(3)}{5-3} = \frac{425-207}{2} = \frac{218}{2} = 109 \text{ meters/min}$

$s(5) = 5^3 + 60(5) = 125 + 300 = 425$

$s(3) = 3^3 + 60(3) = 27 + 180 = 207$

(c) Fill out the tables below and be sure not to round too much.

Time interval	$[3, 3.5]$	$[3, 3.1]$	$[3, 3.01]$	$[3, 3.001]$
Change in time (Δt)	0.5	0.1	0.01	0.001
Change in distance (Δs)	s(3.5) − s(3) 45.875	s(3.1) − s(3) 8.791	s(3.01) − s(3) 0.870901	s(3.001) − s(3) 0.087009001
Average velocity ($\frac{\Delta s}{\Delta t}$)	91.75	87.91	87.0901	87.009001

Time interval	$[2.5, 3]$	$[2.9, 3]$	$[2.99, 3]$	$[2.999, 3]$
Change in time (Δt)	0.5	0.1	0.01	0.001
Change in distance (Δs)	s(3) − s(2.5) 41.375	s(3) − s(2.9) 8.611	s(3) − s(2.99) 0.869101	s(3) − s(2.999) 0.086991001
Average velocity ($\frac{\Delta s}{\Delta t}$)	82.75	86.11	86.9101	86.991001

(d) Using your tables above, make a conjecture about the instantaneous velocity of the boat at 3 minutes into its trip.

The boat seems to be going 87 meters/min at $t=3$ minutes.

3. (This problem spans two pages.) The following questions pertain to the limit definition of the derivative either at a single point c or the derivative function. In each question below, **you must use the specified limit definition to compute the derivative to receive full credit.**

   (a) Using the limit definition of the derivative at a point c, namely $f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$, compute the derivative of $f(x) = 2x^2 + x$ at $c = 1$.

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

$c = 1$

$$f'(1) = \lim_{x \to 1} \frac{f(x) - 3}{x - 1}$$

$f(x) = 2x^2 + x$

$$= \lim_{x \to 1} \frac{2x^2 + x - 3}{x - 1}$$

$$= \lim_{x \to 1} \frac{(2x+3)(x-1)}{x-1}$$

$$= \lim_{x \to 1} 2x + 3$$

$$= 2(1) + 3$$

$$= 5$$

$$\boxed{f'(1) = 5}$$

   (b) Using the limit definition of the derivative at a point c, namely $f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$, compute the derivative of $f(x) = \sqrt{2x}$ at $c = 2$.

$$f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

$c = 2$

$$f'(2) = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h}$$

$f(x) = \sqrt{2x}$

$$= \lim_{h \to 0} \frac{\sqrt{2(2+h)} - \sqrt{2(2)}}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{4+2h} - \sqrt{4}}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{4+2h} - 2}{h} \left(\frac{\sqrt{4+2h} + 2}{\sqrt{4+2h} + 2} \right)$$

$$= \lim_{h \to 0} \frac{4 + 2h - 4}{h(\sqrt{4+2h} + 2)}$$

$$= \lim_{h \to 0} \frac{2h}{h(\sqrt{4+2h} + 2)}$$

$$= \lim_{h \to 0} \frac{2}{\sqrt{4+2h} + 2}$$

$$= \frac{2}{\sqrt{4+2(0)} + 2}$$

$$= \frac{2}{\sqrt{4} + 2}$$

$$= \frac{2}{2+2}$$

$$= \frac{2}{4}$$

$$= \frac{1}{2}$$

$$\boxed{f'(2) = \frac{1}{2}}$$

(c) Using the limit definition of the derivative function, namely $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$, compute the derivative function of the functions below. Specify, in interval notation, where f is continuous and where f is differentiable.

i. $f(x) = \sqrt{3-x}$

$$f'(x) = \lim_{h\to 0} \frac{\sqrt{3-(x+h)} - \sqrt{3-x}}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{3-x-h} - \sqrt{3-x}}{h}\left(\frac{\sqrt{3-x-h} + \sqrt{3-x}}{\sqrt{3-x-h} + \sqrt{3-x}}\right)$$

$$= \lim_{h\to 0} \frac{(3-x-h) - (3-x)}{h(\sqrt{3-x-h} + \sqrt{3-x})}$$

$$= \lim_{h\to 0} \frac{3-x-h-3+x}{h(\sqrt{3-x-h} + \sqrt{3-x})}$$

$$= \lim_{h\to 0} \frac{-h}{h(\sqrt{3-x-h} + \sqrt{3-x})}$$

$$= \lim_{h\to 0} \frac{-1}{\sqrt{3-x-h} + \sqrt{3-x}}$$

$$= \frac{-1}{\sqrt{3-x} + \sqrt{3-x}}$$

$$\boxed{f'(x) = -\frac{1}{2\sqrt{3-x}}}$$

Continuity:

$f(x) = \sqrt{3-x}$ is continuous on its domain. Hence, f is continuous on $(-\infty, 3]$

Differentiability:

$f(x)$ is differentiable wherever $f'(x) = \frac{1}{2\sqrt{3-x}}$ is continuous. Hence, $f(x)$ is differentiable on the domain of $f'(x)$, which is $(-\infty, 3)$.

ii. $f(x) = -\frac{5}{x^2}$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{-\frac{5}{(x+h)^2} - \frac{-5}{x^2}}{h}$$

$$= \lim_{h\to 0} \frac{1}{h}\left[-\frac{5}{(x+h)^2} + \frac{5}{x^2}\right]$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{-5x^2 + 5(x+h)^2}{x^2(x+h)^2}\right]$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{-5x^2 + 5x^2 + 10xh + 5h^2}{x^2(x+h)^2}\right]$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{10xh + 5h^2}{x^2(x+h)^2}\right]$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{h(10x+5h)}{x^2(x+h)^2}\right]$$

$$= \lim_{h\to 0} \frac{10x+5h}{x^2(x+h)^2}$$

$$= \frac{10x}{x^2(x)^2}$$

$$= \frac{10x}{x^4}$$

$$\boxed{f'(x) = \frac{10}{x^3}}$$

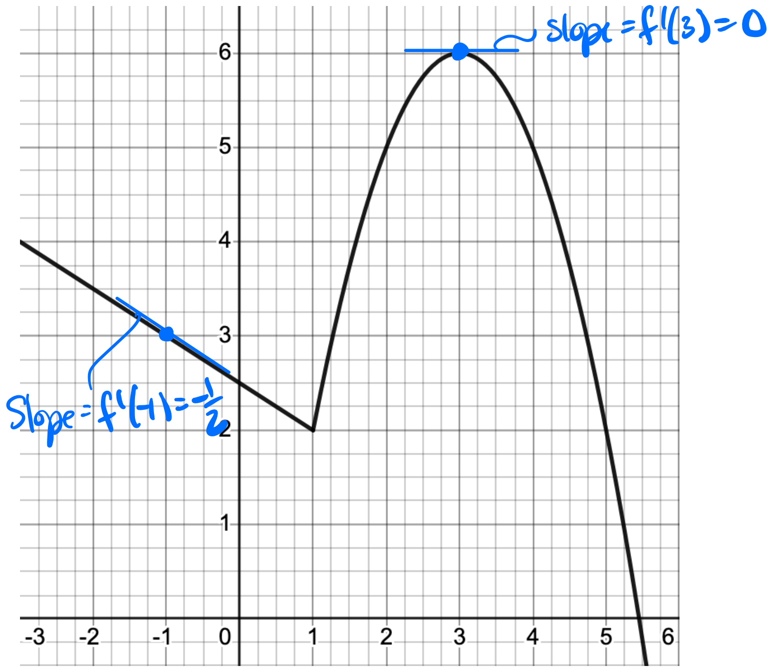
Continuous:

f is continuous on $(-\infty, 0) \cup (0, \infty)$

Differentiable:

f is differentiable on $(-\infty, 0) \cup (0, \infty)$

4. Consider the piecewise function $f(x)$ given below with its graph. Answer parts $(a) - (d)$ simply by looking at the graph.



$$f(x) = \begin{cases} -\frac{1}{2}x + \frac{5}{2} & \text{if } x \leq 1 \\ -x^2 + 6x - 3 & \text{if } x > 1 \end{cases}$$

(a) $f(-1) =$ 3

(b) $f'(-1) = -\frac{1}{2}$ (TL slope is same as linear function)

(c) $f(3) =$ 6

(d) $f'(3) =$ 0 (TL is horizontal)

(e) Notice from the graph that $f(x)$ is continuous at $x = 1$. Show that $f(x)$ is not differentiable $x = 1$ by **explicitly computing the left- and right-side limits of the derivative**.

$$f'(1) = \lim_{x \to 1} \frac{f(x) - f(1)}{x - 1} = \text{DNE}$$

$$\lim_{x \to 1^-} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1^-} \frac{-\frac{1}{2}x + \frac{5}{2} - 2}{x - 1} = \lim_{x \to 1^-} \frac{-\frac{1}{2}x + \frac{1}{2}}{x - 1} = \lim_{x \to 1^-} \frac{-\frac{1}{2}(x-1)}{x-1} = \lim_{x \to 1^-} -\frac{1}{2} = -\frac{1}{2}$$

$$\lim_{x \to 1^+} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1^+} \frac{-x^2 + 6x - 3 - 2}{x - 1} = \lim_{x \to 1^+} \frac{-x^2 + 6x - 5}{x - 1} = \lim_{x \to 1^+} \frac{-(x^2 - 6x + 5)}{x - 1} = \lim_{x \to 1^+} \frac{-(x-5)(x-1)}{x-1} = \lim_{x \to 1^+} -(x-5) = -(1-5) = -(-4) = 4$$

not the same

5. Application Problem: The owner of a small toy manufacturer has determined that he can sell x toys if the price is $D(x) = 0.2x + 30$ dollars. The total cost as a function of x is given by $C(x) = 0.1x^2 + 15x + 247.5$ dollars. (Hint: to do this problem, you might have to *do some light research* about price, cost, and profit functions.)

   (a) Find any break-even points. (Hint: you have to find the *revenue* function first.)

Price: $D(x) = 0.2x + 30$

Revenue: $R(x) = x\,D(x) = 0.2x^2 + 30x$

Costs: $C(x) = 0.1x^2 + 15x + 247.5$

Break-even occurs when $R(x) = C(x)$:

$$0.2x^2 + 30x = 0.1x^2 + 15x + 247.5$$
$$\Rightarrow 0.1x^2 + 15x - 247.5 = 0$$
$$\Rightarrow x^2 + 150x - 2475 = 0$$
$$\Rightarrow (x+165)(x-15) = 0$$
$$\Rightarrow x = -165 \quad \boxed{x = 15}$$

Break-even occurs when we sell $x = 15$ toy soldiers.

   (b) Find the profit function, $P(x)$, and submit a Desmos graph of this function. What do the zeros of this function represent?

$$P(x) = R(x) - C(x) = (0.2x^2 + 30x) - (0.1x^2 + 15x + 247.5)$$
$$\boxed{P(x) = 0.1x^2 + 15x - 247.5}$$

The zeros of this function are the break-even points.

   (c) Compute the marginal profit function using the limit definition of the derivative.

Marginal Profit is $P'(x)$:

$$P'(x) = \lim_{h \to 0} \frac{P(x+h) - P(x)}{h}$$
$$= \lim_{h \to 0} \frac{(0.1(x+h)^2 + 15(x+h) - 247.5) - (0.1x^2 + 15x - 247.5)}{h}$$
$$= \lim_{h \to 0} \frac{0.1x^2 + 0.2xh + 0.1h^2 + 15x + 15h - 247.5 - 0.1x^2 - 15x + 247.5}{h}$$
$$= \lim_{h \to 0} \frac{0.2xh + 0.1h^2 + 15h}{h}$$
$$= \lim_{h \to 0} \frac{h(0.2x + 0.1h + 15)}{h}$$
$$= \lim_{h \to 0} 0.2x + 0.1h + 15$$
$$\boxed{P'(x) = 0.2x + 15}$$