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Instructions: All solutions should be prepared carefully and neatly. All solution sets shall be completed on this packet and submitted by uploading a scan or picture of your written work to D2L by 11:59 PM on the due date below. **Submit only a single pdf file of your entire packet. Submit any Excel or Python files as well.** The mobile app called *Genius Scan* works well. Use a PENCIL and if you make a mistake, use an eraser. Careless presentation (e.g. bad handwriting, pen scribbles, doodles, wasted space, etc) will result in a deduction of points at my discretion. Submitted work that does not demonstrate clearly the process by which one arrived at the answer will not receive credit of any kind. Academic dishonesty will not be tolerated.

Problem Set II

MAT 362-010 – Operations Research II

Due: Friday, March 1 by 11:59 PM on D2L

Read: Sections 6.7–6.10, 7.1–7.3

Solutions!

Problem Number	Available Points	Your Points
1	5	5
2	5	5
3	5	5
4	5	5
5	5	5
6	5	5
7	5	5
8	5	5
Total	40	40

1. Consider the following LP: [(5)]

$$\text{Maximize:} \quad z = 5x_1 + 2x_2 + 3x_3$$

$$\begin{aligned} \text{Subject to:} \quad & x_1 + 5x_2 + 2x_3 = 30 \\ & x_1 - 5x_2 - 6x_3 \le 40 \\ & x_1, x_2, x_3 \ge 0 \end{aligned}$$

Suppose the optimal tableau for this problem is below and that $M = 100$ when solving the problem.

Row	Basic	z	x_1	x_2	x_3	a_1	s_2	RHS
0	z	1	0	23	7	105	0	150
1	x_1	0	1	5	2	1	0	30
2	s_2	0	0	-10	-8	-1	1	10

Write the associated dual problem and determine its optimal solution in two different ways by observation above and by computing $\boldsymbol{c}_{BV}B^{-1}$.

Dual LP:

Minimize $w = 30y_1 + 40y_2$

Subject to

$$\begin{aligned} y_1 + y_2 &\ge 5 \\ 5y_1 - 5y_2 &\ge 2 \\ 2y_1 - 6y_2 &\ge 3 \\ y_1 \text{ urs}, \ & y_2 \ge 0 \end{aligned}$$

Solution 1: Read from Row 0

$y_1 = 105 - 100 = 5$

$y_2 = 0$

$$\boxed{\vec{y} = [5 \quad 0]}$$

Solution 2:

$$\vec{y} = \vec{c}_{BV}^{\,T} B^{-1} = [5 \quad 0]\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} = [5+0 \quad 0+0] = [5 \quad 0]$$

$$\boxed{\vec{y} = [5 \quad 0]}$$

2. Consider the following LP: [(5)]

$$\text{Minimize:} \quad z = 5x_1 + 2x_2$$

$$\begin{aligned} \text{Subject to:} \quad x_1 - x_2 &\geq 3 \\ 2x_1 + 3x_2 &\geq 5 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Dual LP:

Maximize $\omega = 3y_1 + 5y_2$

Subject to $y_1 + 2y_2 \leq 5$

$-y_1 + 3y_2 \leq 2$

$y_1, y_2 \geq 0$

Determine whether or not the following pairs of primal-dual solutions are optimal:

a.) Primal: $x_1 = 3$, $x_2 = 1$. Dual: $y_1 = 4$, $y_2 = 1$

Primal: Constraint 1: $3 - 1 \not\geq 3$

Not feasible

Dual: Constraint 1: $4 + 2(1) \not\leq 5$

Not feasible

Neither solution is feasible.

b.) Primal: $x_1 = 4$, $x_2 = 1$. Dual: $y_1 = 1$, $y_2 = 0$

Both solutions are feasible, but

$$z = 5(4) + 2(1) = 22 \neq 3 = 3(1) + 5(0) = \omega$$

We have that $\omega < z$. Thus, the solutions are not optimal.

c.) Primal: $x_1 = 3$, $x_2 = 0$. Dual: $y_1 = 5$, $y_2 = 0$

Both solutions are feasible, and

$$z = 5(3) + 2(0) = 15 = 3(5) + 5(0) = \omega$$

Hence, both solutions are optimal

$$\omega_{opt} = 15 = z_{opt}.$$

3. Consider the following LP: [(5)]

$$\text{Minimize:} \quad z = 2x_1 + x_2$$

$$\begin{aligned} \text{Subject to:} \quad 3x_1 + x_2 &\geq 3 \\ 4x_1 + 3x_2 &\geq 6 \\ x_1 + 2x_2 &\leq 3 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Standard Form:

$$z = 2x_1 + x_2 + Ma_1 + Ma_2$$

$$\rightarrow \begin{aligned} 3x_1 + x_2 - e_1 + a_1 &= 3 \\ 4x_1 + 3x_2 - e_2 + a_2 &= 6 \\ x_1 + 2x_2 + s_3 &= 3 \end{aligned}$$

Suppose x_1, x_2, and s_3 are a set of basic variables and the associated inverse matrix is

$$B^{-1} = \begin{bmatrix} \frac{3}{5} & -\frac{1}{5} & 0 \\ -\frac{4}{5} & \frac{3}{5} & 0 \\ 1 & -1 & 1 \end{bmatrix}.$$

Using this inverse matrix, construct the entire simplex tableau (below) associated with this set of variables and check it for feasibility and optimality. (Use the space below to show your work.)

- Dual Solution: $\vec{y} = \vec{C}_{BV}^T B^{-1} = [2 \ 1 \ 0] \begin{bmatrix} 3/5 & -1/5 & 0 \\ -4/5 & 3/5 & 0 \\ 1 & -1 & 1 \end{bmatrix} = [2/5 \ \ 1/5 \ \ 0]$

- Primal Solution: $\vec{x} = B^{-1}\vec{b} = \begin{bmatrix} 3/5 & -1/5 & 0 \\ -4/5 & 3/5 & 0 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 6 \\ 3 \end{bmatrix} = \begin{bmatrix} 9/5 - 6/5 \\ -12/5 + 18/5 \\ 3 - 6 + 3 \end{bmatrix} = \begin{bmatrix} 3/5 \\ 6/5 \\ 0 \end{bmatrix}$

- Dual Obj. Function Value: $w = 3y_1 + 6y_2 + 3y_3 = 3(2/5) + 6(1/5) + 3(0) = 6/5 + 6/5 = 12/5$

- Primal Obj. Function Value: $z = 2x_1 + x_2 = 2(3/5) + 6/5 = 6/5 + 6/5 = 12/5$

Feasible and $w_{opt} = z_{opt}$ ↳ Optimal!

- $B^{-1}\vec{a}_3 = \begin{bmatrix} 3/5 & -1/5 & 0 \\ -4/5 & 3/5 & 0 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -3/5 \\ 4/5 \\ -1 \end{bmatrix}$

- $B^{-1}\vec{a}_5 = \begin{bmatrix} 3/5 & -1/5 & 0 \\ -4/5 & 3/5 & 0 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/5 \\ -3/5 \\ 1 \end{bmatrix}$

- $B^{-1}\vec{a}_4 = \begin{bmatrix} 3/5 & -1/5 & 0 \\ -4/5 & 3/5 & 0 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 3/5 \\ -4/5 \\ 1 \end{bmatrix}$

- $B^{-1}\vec{a}_6 = \begin{bmatrix} 3/5 & -1/5 & 0 \\ -4/5 & 3/5 & 0 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1/5 \\ 3/5 \\ -1 \end{bmatrix}$

- Row 0: $C_{BV}^T B^{-1} [\vec{a}_3 \ \vec{a}_4 \ \vec{a}_5 \ \vec{a}_6] - [0 \ M \ 0 \ M] = [-2/5 \ \ 2/5 - M \ \ -1/5 \ \ 1/5 - M]$

Row	Basic	x_1	x_2	e_1	a_1	e_2	a_2	s_3	RHS
0	z	0	0	$-2/5$	$2/5 - M$	$-1/5$	$1/5 - M$	0	$12/5$
1	x_1	1	0	$-3/5$	$3/5$	$1/5$	$-1/5$	0	$3/5$
2	x_2	0	1	$4/5$	$-4/5$	$-3/5$	$3/5$	0	$6/5$
3	s_3	0	0	-1	1	1	-1	1	0

4. Consider the following LP: [(5)]

$$\text{Maximize:} \quad z = 3x_1 + 2x_2 + 5x_3$$

$$\begin{aligned} \text{Subject to:} \quad & x_1 + 2x_2 + x_3 \le 30 + \Delta b_1 \\ & 3x_1 + 2x_3 \le 60 + \Delta b_2 \\ & x_1 + 4x_2 \le 20 + \Delta b_3 \\ & x_1, x_2, x_3 \ge 0 \end{aligned}$$

Show that the set x_2, x_3, and s_3 is a set optimal variables using the associated inverse matrix:

$$B^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{4} & 0 \\ 0 & \frac{1}{2} & 0 \\ -2 & 1 & 1 \end{bmatrix}.$$

Use this matrix to determine the feasibility ranges of the right hand side values of each constraint.

New Solution:

$$\vec{x} = \begin{bmatrix} 1/2 & -1/4 & 0 \\ 0 & 1/2 & 0 \\ -2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 30+\Delta b_1 \\ 60+\Delta b_2 \\ 20+\Delta b_3 \end{bmatrix} = \begin{bmatrix} 15+\frac{1}{2}\Delta b_1 - 15 - \frac{1}{4}\Delta b_2 \\ 30+\frac{1}{2}\Delta b_2 \\ -60-2\Delta b_1+60+\Delta b_2+20+\Delta b_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\Delta b_1 - \frac{1}{4}\Delta b_2 \\ 30+\frac{1}{2}\Delta b_2 \\ 20-2\Delta b_1+\Delta b_2+\Delta b_3 \end{bmatrix}$$

Feasibility Range for b_1: Assume $\Delta b_2 = \Delta b_3 = 0$, then

$$\begin{aligned} \tfrac{1}{2}\Delta b_1 &\ge 0 \\ 20 - 2\Delta b_1 &\ge 0 \end{aligned} \Rightarrow \begin{aligned} \Delta b_1 &\ge 0 \\ \Delta b_1 &\le 10 \end{aligned} \Rightarrow \boxed{30 \le b_1 \le 40}$$

Feasibility Range for b_2: Assume $\Delta b_1 = \Delta b_3 = 0$, then

$$\begin{aligned} -\tfrac{1}{4}\Delta b_2 &\ge 0 \\ 30 + \tfrac{1}{2}\Delta b_2 &\ge 0 \\ 20 + \Delta b_2 &\ge 0 \end{aligned} \Rightarrow \begin{aligned} \Delta b_2 &\le 0 \\ \Delta b_2 &\ge -60 \\ \Delta b_2 &\ge -20 \end{aligned} \Rightarrow \boxed{40 \le b_2 \le 60}$$

Feasibility Range for b_3: Assume $\Delta b_1 = \Delta b_2 = 0$, then

$$20 + \Delta b_3 \ge 0 \Rightarrow \Delta b_3 \ge -20 \Rightarrow \boxed{b_3 \ge 0}$$

5. A company supplies goods to three customers, who each require 30 units. The company has two warehouses. Warehouse 1 has 40 units available, and warehouse 2 has 30 units available. The costs of shipping 1 unit from warehouse to customer are shown in the table below. There is a penalty for each unmet customer unit of demand: With customer 1, a penalty case of \$90 is incurred; with customer 2, \$80; and with customer 3, \$110. Formulate a balanced transportation problem (i.e. create a transportation grid) to minimize the sum of shortage and shipping costs. You do not have to solve it. [(5)]

	To		
From	Customer 1	Customer 2	Customer 3
Warehouse 1	\$15	\$35	\$25
Warehouse 2	\$10	\$50	\$40

Supply: warehouse 1 : 40
warehouse 2 : 30 } 70

Demand: Customer 1 : 30
Customer 2 : 30 } 90
Customer 3 : 30

} unbalanced ⇒ Need a dummy warehouse.

Costs are penalties

Formulation:

	Customer 1	Customer 2	Customer 3	Supply
Warehouse 1	15	35	25	40
Warehouse 2	10	50	40	30
(Shortage) Dummy	90	80	110	20
Demand:	30	30	30	

6. A bank has two sites at which checks are processed. Site 1 can process 10,000 checks per day and site 2 can process 6,000 checks per day. The bank processes three types of checks: vendor, salary, and personal. The processing cost per check depends on the site, see the table below. Each day, 5,000 checks of each type must be processed. Formulate and solve in Excel the balanced transportation problem to minimize the daily cost of processing checks. [(5)]

Check	Site 1	Site 2
Vendor	\$0.05	\$0.03
Salary	\$0.04	\$0.04
Personal	\$0.02	\$0.05

Supply: (sites) Site 1 : 10,000
Site 2 : 6,000 } 16,000

Demand: (check type) Vendor : 5,000
Salary : 5,000
Personal : 5,000 } 15,000

unbalanced ⇒ needs a dummy demand point.
costs are zero

Formulation:

	Vendor	Salary	Personal	Dummy	Supply:
Site 1	5	4 / 4000	2 / 5000	0 / 1000	10,000
Site 2	3 / 5000	4 / 1000	5	0	6,000
Demand:	5,000	5,000	5,000	1,000	

From Excel: $x_{12} = 4000$, $x_{13} = 5000$, $x_{14} = 1000$
$x_{21} = 5000$, $x_{22} = 1000$

Cost: $z = 450$ (surplus at site 1)

Key

North-west Corner

Min Cost Method

Vogel's Method

7. For each of the transportation models below, construct a starting bfs using the Norhwest-Corner, Minimum-Cost, and Vogel's method. For each model and method, compute the initial cost. [(5)]

a.)

Northwest:

				Supply:
	5 [0]	1 [2]	[1]	6
	[2]	4 [1]	3 [5]	7
	[2]	[4]	7 [3]	7
Demand:	5	5	10	

Cost = 42

Min Cost:

				Supply:
	5 [0]	[2]	1 [1]	6
	[2]	5 [1]	2 [5]	7
	[2]	[4]	7 [3]	7
Demand:	5	5	10	

Cost = 37

Vogels:

				Supply:
	5 [0]	[2]	1 [1]	6
	[2]	5 [1]	2 [5]	7
	[2]	[4]	7 [3]	7
Demand:	5	5	10	

Cost = 37

b.)

Northwest:

				Supply:
	7 [1]	[2]	[6]	7
	3 [0]	9 [4]	[2]	12
	[3]	1 [1]	10 [5]	11
Demand:	10	10	10	

Cost = 94

Min Cost:

				Supply:
	[1]	[2]	7 [6]	7
	10 [0]	[4]	2 [2]	12
	[3]	10 [1]	1 [5]	11
Demand:	10	10	10	

Cost = 61

Vogels:

				Supply:
	7 [1]	[2]	[6]	7
	2 [0]	[4]	10 [2]	12
	1 [3]	10 [1]	[5]	11
Demand:	10	10	10	

Cost = 40

c.)

Northwest:

				Supply:
	9 [5]	3 [1]	[8]	12
	[2]	7 [4]	7 [0]	14
	[3]	[6]	4 [7]	4
Demand:	9	10	11	

Cost = 104

Min Cost:

				Supply:
	2 [5]	10 [1]	[8]	12
	3 [2]	[4]	11 [0]	14
	4 [3]	[6]	[7]	4
Demand:	9	10	11	

Cost = 38

Vogels:

				Supply:
	2 [5]	10 [1]	[8]	12
	3 [2]	[4]	11 [0]	14
	4 [3]	[6]	[7]	4
Demand:	9	10	11	

Cost = 38

8. Solve Problem 5 using the Transportation Simplex Method. Start with Northwest Method [(5)]

	$v_1=15$	$v_2=35$	$v_3=25$
$u_1=0$	[15] 30	[35] 10	[25] 0
$u_2=15$	[10] 20	[50] 20	[40] 10
$u_3=85$	[90] 10	* [80] 40	[10] 20

→

[15] 30	[35] 10	[25]
[10]	[50] 20-θ	[40] 10+θ
[90]	[80] θ	[10] 20-θ

	$v_1=15$	$v_2=35$	$v_3=65$
$u_1=0$	[15] 30	[35] 10	* [25] 40
$u_2=-15$	[10] -10	[50] -30	[40] 30
$u_3=45$	[90] -30	[80] 20	[10] 0

→

[15] 30	[35] 10-θ	[25] θ
[10]	[50]	[40] 30
[90]	[80] 20+θ	[10] 0-θ

	$v_1=15$	$v_2=35$	$v_3=25$
$u_1=0$	[15] 30	[35] 10	[25] 0
$u_2=15$	* [10] 20	[50] 0	[40] 30
$u_3=45$	[90] -30	[80] 20	[10] -40

→

[15] 30-θ	[35] 10	[25] 0+θ
[10] θ	[50]	[40] 30-θ
[90]	[80] 20	[10]

	$v_1 = 15$	$v_2 = 35$	$v_3 = 25$
$u_1 = 0$	$\boxed{15}$ 0	$\boxed{35}$ 10	$\boxed{25}$ 30
$u_2 = -5$	$\boxed{10}$ 30	$\boxed{50}$ -20	$\boxed{40}$ -20
$u_3 = 45$	$\boxed{90}$ -30	$\boxed{80}$ 20	$\boxed{10}$ -40

Optimal! $Z_{opt} = 3000$