By signing below, you attest that you have neither given nor received help of any kind on this exam.

Signature: Brooks Emerick Printed Name: Brooks Emerick

Instructions: Show work to get full credit (the correct answer may NOT be enough). Do all your work on the paper provided. Write clearly! Double check your answers!

You will **not** receive full credit for using methods other than those discussed in class.

Calculators are not permitted.

Exam I

MAT 340 – Differential Equations

Solutions!

Problem Number	Available Points	Your Points
(2) 1	6	6
(2) 2	6	6
(3.5) 3	8	8
(2) 4	7	7
(5) 5	7	7
(1) 6	4	4
Total	38	38

(15.5)

1. Consider the following differential equation: [(6)]

$$\frac{dy}{dt} + y = 10\cos(3t)$$

(a) Find the general solution to this equation. [5]

H: $\frac{dy}{dt} = -y \longrightarrow y_h(t) = e^{-t}$

N-H: Guess: $y_p(t) = a\cos(3t) + b\sin(3t)$

$$\frac{dy_p}{dt} + y_p = -3a\sin(3t) + 3b\cos(3t) + a\cos(3t) + b\sin(3t)$$
$$= (a+3b)\cos(3t) + (-3a+b)\sin(3t)$$

$$\Rightarrow \begin{cases} a+3b=10 \\ -3a+b=0 \end{cases} \Rightarrow \begin{cases} a+9a=10 \\ b=3a \end{cases} \Rightarrow \begin{cases} a=1 \\ b=3 \end{cases}$$

$$y_p(t) = \cos(3t) + 3\sin(3t)$$

$$\boxed{y(t) = ke^{-t} + \cos(3t) + 3\sin(3t)}$$

(b) Consider the word bank below. Circle all the words that describe this differential equation. [1]

autonomous (ordinary) partial homogeneous (non-homogeneous) (linear)

non-linear harmonic oscillator (first-order) second-order separable

2. Consider the following first-order ODE: [(6)]

$$\frac{dy}{dt} = -\frac{y}{t} + e^{-t^2} + t$$

(a) Find the general solution to this ODE. [5]

$\frac{dy}{dt} + \frac{1}{t}y = e^{-t^2} + t$ $\quad \mu(t) = e^{\int \frac{1}{t}dt} = e^{\ln(t)} = t$

$\Rightarrow \quad t\frac{dy}{dt} + y = te^{-t^2} + t^2$

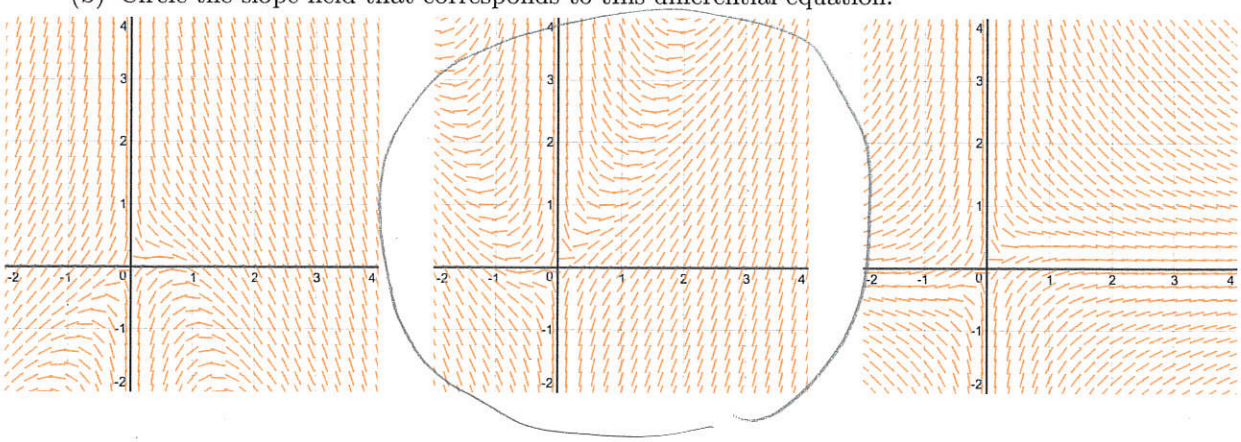
$\Rightarrow \quad \frac{d}{dt}(ty) = te^{-t^2} + t^2$

$\Rightarrow \quad ty = \int te^{-t^2} + t^2\,dt$ $\quad u = -t^2$, $du = -2t\,dt$

$\Rightarrow \quad y(t) = \frac{1}{t}\left(-\frac{1}{2}e^{-t^2} + \frac{1}{3}t^3 + C\right)$

$\Rightarrow \quad \boxed{y(t) = -\frac{1}{2t}e^{-t^2} + \frac{1}{3}t^2 + \frac{C}{t}}$

(b) Circle the slope field that corresponds to this differential equation: [1]



3. Consider the following initial value problem: [(8)]

$$\frac{dy}{dt} = 3\sqrt{y(t-1)} \qquad y(2) = 0$$

(a) Determine, with proper justification, if a solution to the IVP exists. [1]

$f(t,y) = 3\sqrt{y}\sqrt{t-1}$ which is continuous for all $t \in [1,\infty)$ and $y \in [0,\infty)$. Since f is continuous at $t_0 = 2$ and $y_0 = 0$, we know a solution exists.

(b) Use separation of variables to solve the IVP. [5]

$$\frac{dy}{dt} = 3\sqrt{y}\sqrt{t-1}$$

$$\Rightarrow \frac{1}{\sqrt{y}}\,dy = 3\sqrt{t-1}\,dt$$

$$\Rightarrow \int y^{-1/2}\,dy = 3\int (t-1)^{1/2}\,dt$$

$$\Rightarrow 2y^{1/2} = 3\left(\tfrac{2}{3}\right)(t-1)^{3/2} + C$$

$$\Rightarrow y^{1/2} = (t-1)^{3/2} + C$$

$$\Rightarrow y(t) = \left[(t-1)^{3/2} + C\right]^2$$

$$\Rightarrow y(2) = \left[(2-1)^{3/2} + C\right]^2$$

$$\Rightarrow 0 = (C+1)^2$$

$$\Rightarrow C = -1$$

$$\boxed{y(t) = \left[(t-1)^{3/2} - 1\right]^2}$$

(c) Is the solution you found to part b unique? Justify your answer appropriately. [2]

$\frac{\partial f}{\partial y} = \frac{3\sqrt{t-1}}{2\sqrt{y}}$ which is not continuous at $y_0 = 0$. Hence, our solution may not be unique. In fact, it isn't because $y(t) = 0$ solves the IVP as well.

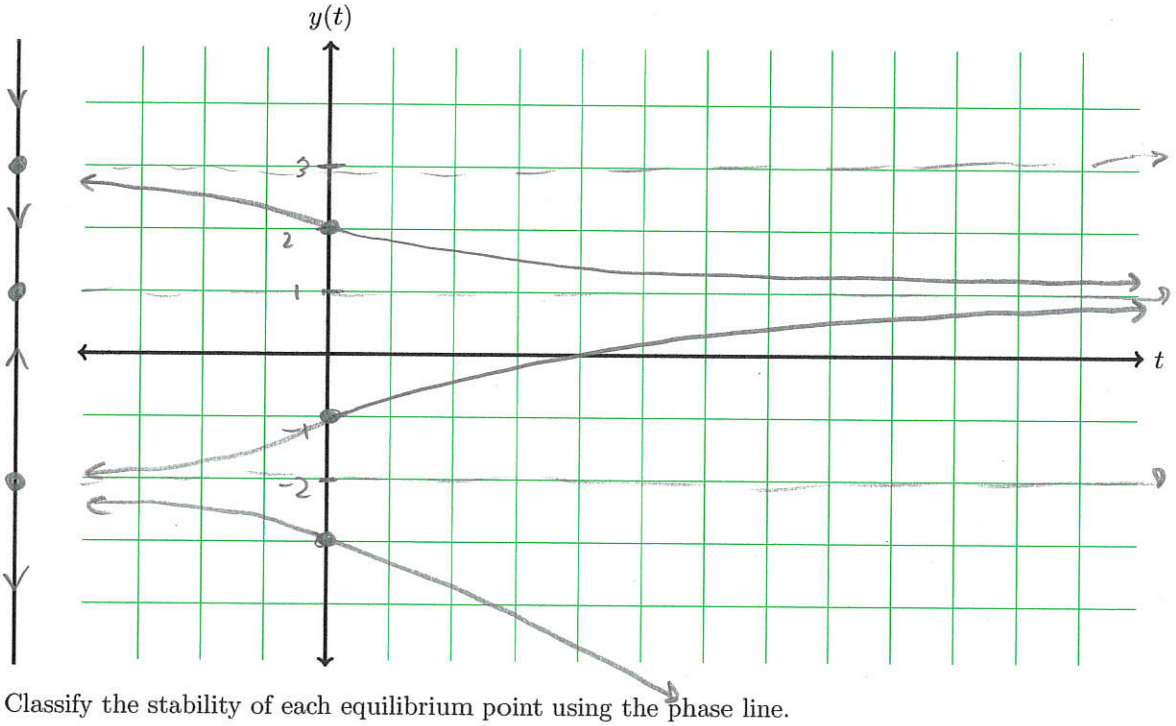
4. Consider the following autonomous ODE: [(7)]

$$\frac{dy}{dt} = (1-y)(y+2)(y-3)^2$$

(a) Find all the equilibria of this differential equation. [1]

$y_1^* = -2$, $y_2^* = 1$, $y_3^* = 3$

(b) Create a phase line on the vertical line to the left of the graph below. On the grid paper provided, sketch the solutions with initial conditions $y(0) = -1$, $y(0) = 2$, and $y(0) = -3$. [3]



(c) Classify the stability of each equilibrium point using the phase line. [2]

$y_1^* = -2$ is a source

$y_2^* = 1$ is a sink

$y_3^* = 3$ is a semi-stable node

(d) Suppose $y(0) = 4$. What is the range of the solution? [1]

Range of $y(t)$ is $(3, \infty)$

5. Consider the following family of ODEs in the real-valued parameter a: [(7)]

$$\frac{dy}{dt} = y^3 - (a-2)y.$$

(a) Find all the equilibria, y^*, in terms of the parameter a. [2]

$0 = y^3 - (a-2)y$

$\Rightarrow y(y^2 - (a-2)) = 0$

$y_1^* = 0, \quad y_2^* = \sqrt{a-2} \quad y_3^* = -\sqrt{a-2}$

(b) Determine the critical bifurcation parameter value(s) and summarize for which values of the parameter a the equilibria are stable, unstable, or semi-stable using the Linearization Theorem. If the Linearization Theorem is inconclusive, use other means to determine stability. [5]

$a^* = 2$ as the number of equilibria changes here.

$f_a'(y) = 3y^2 - (a-2)$

$a < a^*$: One Equilibria $y_1^* = 0$

$f_a'(y_1^*) = f_a'(0) = 3(0)^2 - (a-2) = 2-a > 0 \rightarrow$ Source

$y_1^* = 0$ is a source on $a < 2$

$a = a^*$: Note $f_a(y) = y^3 \rightarrow$ increasing across $y = 0$

One equilibria $y_1^* = 0$

$f_a'(y_1^*) = f_a'(0) = 3(0)^2 - (a-2) = 2-a = 2-2 = 0 \rightarrow$ inconclusive

$y_1^* = 0$ is a source since $f_a(y)$ is increasing

$a > a^*$: Three equilibria $y_1^* = 0, \; y_2^* = \sqrt{a-2}, \; y_3^* = -\sqrt{a-2}$

$f_a'(y_1^*) = f_a'(0) = 2-a < 0 \rightarrow$ Sink

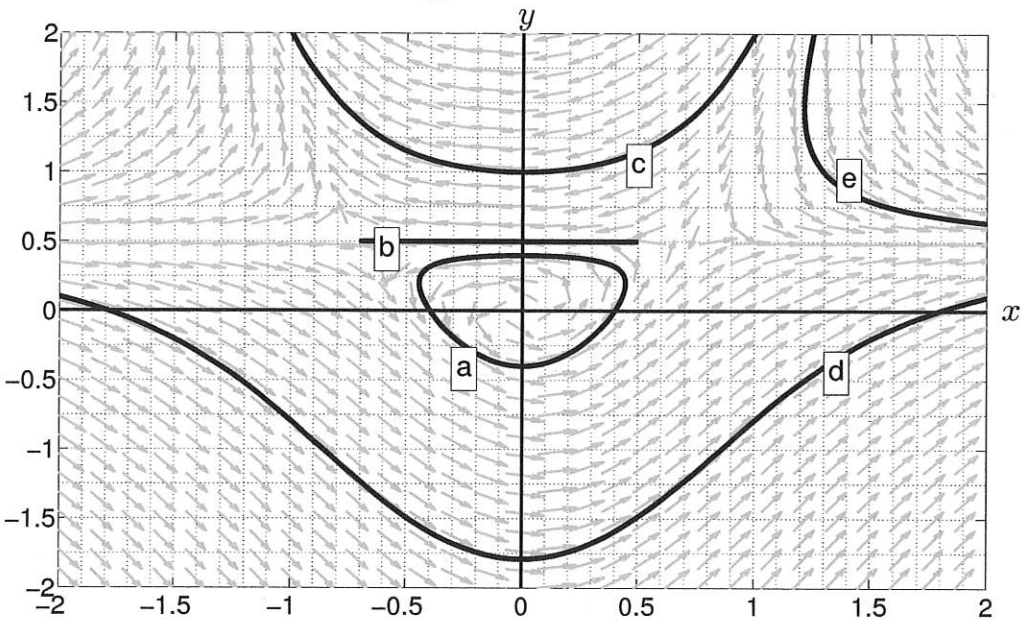
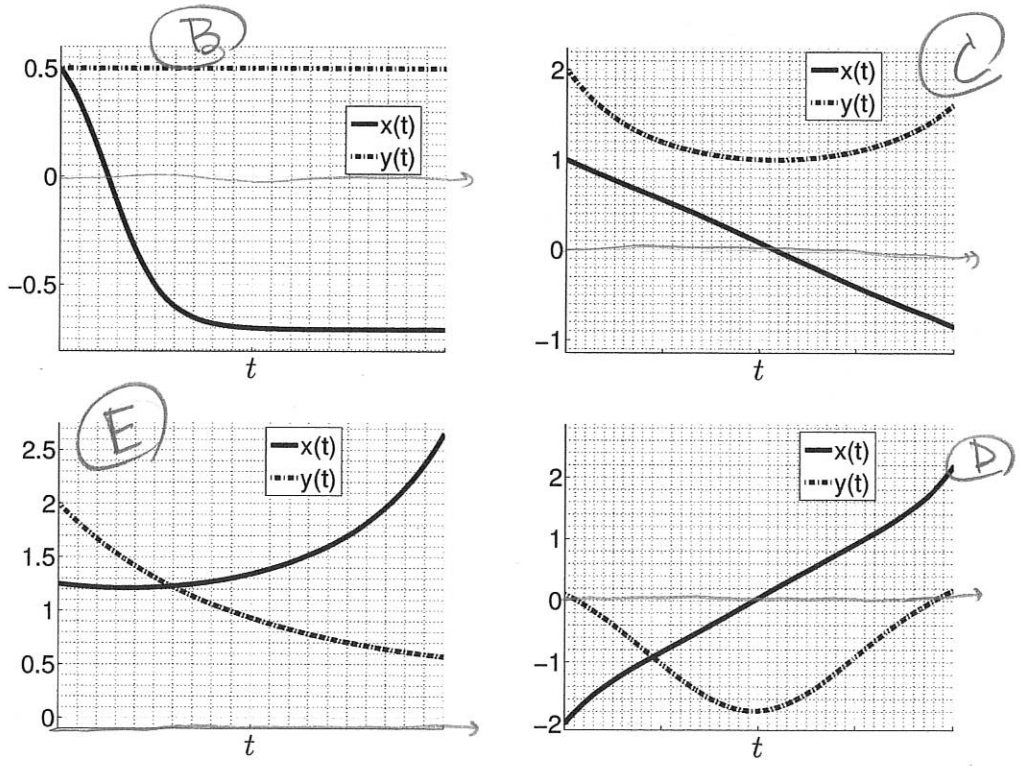
$f_a'(y_2^*) = f_a'(\sqrt{a-2}) = 3(\sqrt{a-2})^2 - (a-2) = 3(a-2) - (a-2) = 2a-4 > 0$ ✓ Sources

$f_a'(y_3^*) = f_a'(-\sqrt{a-2}) = 3(-\sqrt{a-2})^2 - (a-2) = 3(a-2) - (a-2) = 2a-4 > 0$ ✓

y_1^* is a sink, y_2^*, y_3^* are sources

6. The phase portrait containing several solution curves is given on the direction field of the system below: [4]

$$\frac{dx}{dt} = -y + x^2$$
$$\frac{dy}{dt} = x - 2xy$$



Match the solution curve in the direction field above to the corresponding time plot of $x(t)$ and $y(t)$ below by writing the letter of the solution curve next to the plot.