MAT 362 - Operations Research II

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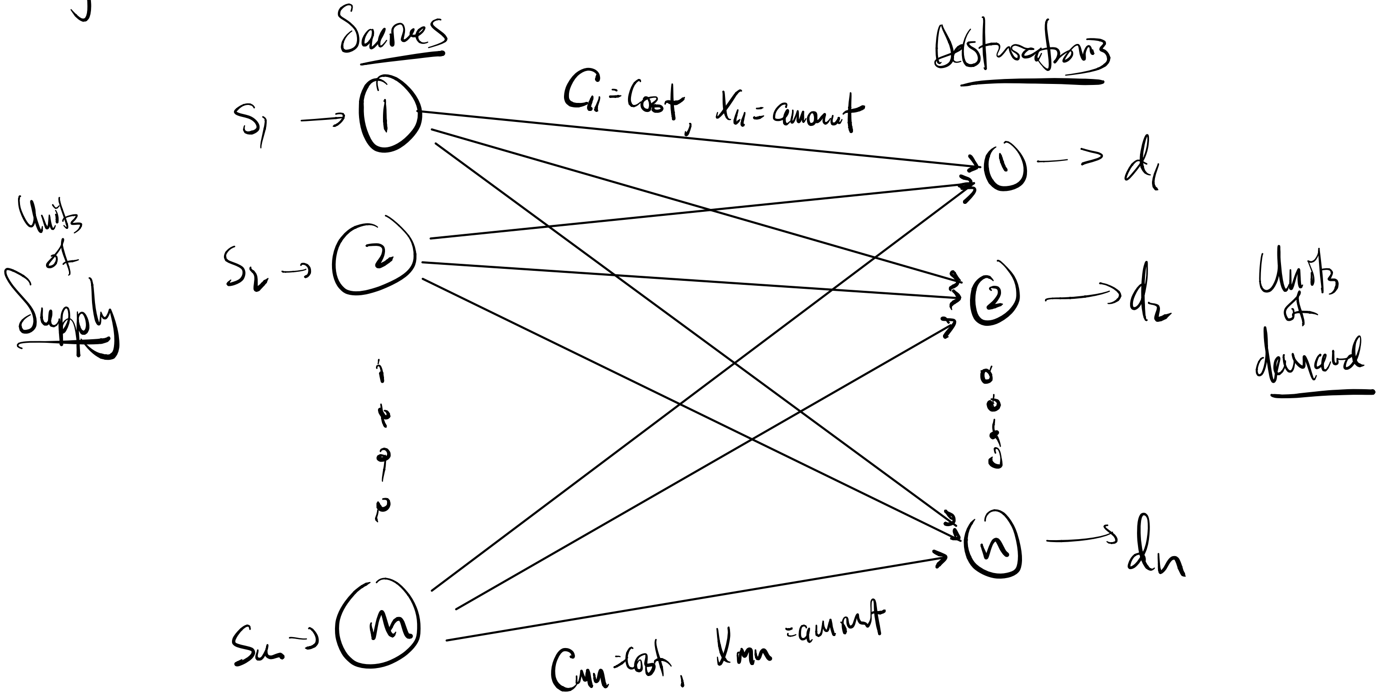
CHAPTER 7 - Transportation Problems

Section 7.1: Formulating Transportation Problems.

- Characteristic of most transportation problems is the idea of a network consisting of m sources and n destinations, each represented by a node. Directed edges represent routes linking sources to destinations.



- We'll start with a basic example.

WS #1 Formulating a transportation LP.

- A balanced Transportation LP occurs when the supply from the sources equals the total demand of the destinations.

- If Supply = Demand → Balanced Transportation LP

  ↳ No changes needed, Solve by Transportation Simplex or using Excel Solver.

- If Supply < Demand → Unbalanced, Shortage will occur at some destination(s).

  ↳
  - We create a Dummy Source.
  - Add a row to the grid with supply equal to the deficit: (Total Demand) − (Total Supply).
  - Adding a row creates the variables $x_{m+1,1}, x_{m+1,2}, \ldots, x_{m+1,n}$.
  - If $x_{m+1,j} > 0$ in the optimal solution, then destination j doesn't receive total demand.
  - To ensure destination j meets demand, use a large cost, M.

- If Supply > Demand → Unbalanced, Surplus will occur at some of the source(s)

  ↳
  - We create a Dummy Destination
  - Add a column to the grid with demand equal to the deficit: (Total Supply) − (Total Demand).
  - Adding a column creates the variables $x_{1,n+1}, x_{2,n+1}, \ldots, x_{m,n+1}$.
  - If $x_{i,n+1} > 0$ in the optimal solution, then source i doesn't ship its full supply.
  - To ensure source i gets rid of entire supply, use a large cost, M.

$\boxed{\text{WS}}$ #2, #3. Formulating and Solving Unbalanced Transportation LPs

$\boxed{\text{WS}}$ #1 Formulating and Solving Transportation LP.

- General Transportation LP:

$$\text{Minimize } z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$$

$$\text{S.T.} \quad \sum_{j=1}^{n} x_{ij} \le s_i \quad \text{for } i = 1, 2, \ldots, m$$

⊜

$$\sum_{i=1}^{m} x_{ij} \ge d_j \quad \text{for } j = 1, 2, \ldots, n.$$

⊜

- In a balanced LP, all constraints become binding — otherwise any remaining supply would be unused leaving a shortage at one of the demand points.

Section 7.2: Finding a bfs to a Transportation LP

- We'll develop a variant of the Simplex Method using the transportation grid as our tableau, called the Transportation Simplex Method. In order to initialize the method, we need a starting bfs.
- There are three popular ways for creating a starting bfs:

1.) Northwest Corner Method – ignores cost

  - Start in cell x_{11}, allocate as much as possible regardless of cost, $x_{11} = \min\{s_1, d_1\}$.
  - Update supply/demand in first row/column.
  - Cross out row/column with zero supply/demand. Break ties arbitrarily.
  - Move to next row (if row was crossed out) or next column (if a column was crossed out). Repeat.

2.) Minimum Cost Method – targets cheapest routes

- Identify cell with minimum cost, allocate maximum amount possible to that cell.
- Update supply/demand in appropriate row/colum.
- Cross out row/column with zero supply/demand. Break ties arbitrarily.
- Move to next available cell with minimum cost. Repeat.

3.) Vogel's Method – improved version of min-cost

- Determine a penalty for each row and column in each iteration:

$$\text{penalty} = \begin{pmatrix}\text{Next smallest} \\ \text{cost}\end{pmatrix} - \begin{pmatrix}\text{smallest} \\ \text{cost}\end{pmatrix}$$

in each row and column.

- Identify the row or column with <u>largest penalty</u>. Use min-cost method on cell will minimum cost in that row/column.
- Cross out row/column with zero supply/demand. Break ties arbitrarily.
- Recalculate penalties for remaining cells in all remaining rows and columns. Repeat.

WS #1–#3 Working with finding a bfs using each method.

Section 7.3: Transportation Simplex Method

- We can use a variant of the Simplex Method by considering the $\overline{c_{ij}}$ value (i.e. the Row 0 coefficient) of the non-basic variables in any iteration.

- The $\bar{c}_{ij}$ value can be quickly calculated using the dual LP.
- Knowing the $\bar{c}_{ij}$ value for every non-basic variable gives us an opportunity to define the <u>entering</u> variable and to determine whether our current basis is optimal.
- We'll demonstrate the dual LP formulation with an example:

<u>Transportation LP</u>:

					Supply
	10	2	20	11	15
	12	7	9	20	25
	4	14	16	18	10
<u>Demand</u>	5	15	15	15	

<u>Transportation Primal LP Formulation</u>:

$$\text{Minimize } Z = 10x_{11} + 2x_{12} + 20x_{13} + 11x_{14} + 12x_{21} + 7x_{22} + \cdots$$
$$9x_{23} + 20x_{24} + 4x_{31} + 14x_{32} + 16x_{33} + 18x_{34}$$

Subject to

$$\begin{array}{lllllllllllll}
x_{11} & +x_{12} & +x_{13} & +x_{14} & & & & & & & & & =15 \\
& & & & x_{21} & +x_{22} & +x_{23} & +x_{24} & & & & & =25 \\
& & & & & & & & x_{31} & +x_{32} & +x_{33} & +x_{34} & =10 \\
x_{11} & & & & +x_{21} & & & & +x_{31} & & & & =5 \\
& x_{12} & & & & +x_{22} & & & & +x_{32} & & & =15 \\
& & x_{13} & & & & +x_{23} & & & & +x_{33} & & =15 \\
& & & x_{14} & & & & +x_{24} & & & & +x_{34} & =15 \\
& & & & & & & & & & & & x_{ij} \geq 0
\end{array}$$

- If we consider the dual problem, we notice that each dual constraint will contain exactly two dual variables: one from the supply constraints and one from the demand constraints.

- When formulating the dual, we define the solution vector $\vec{y}$ as

$$\vec{y} = [u_1 \; u_2 \; \cdots \; u_m \; v_1 \; v_2 \; \cdots \; v_n].$$

Note: $\vec{y}$ is an $1 \times (m+n)$ row vector. Then, in this example, the dual becomes

$$\text{Maximize} \quad w = 15u_1 + 25u_2 + 10u_3 + \cdots$$
$$5v_1 + 15v_2 + 15v_3 + 15v_4$$

Subject to

$$\begin{array}{lll} u_1 + v_1 \le 10 & u_2 + v_1 \le 12 & u_3 + v_1 \le 4 \\ u_1 + v_2 \le 2 & u_2 + v_2 \le 7 & u_3 + v_2 \le 14 \\ u_1 + v_3 \le 20 & u_2 + v_3 \le 9 & u_3 + v_3 \le 16 \\ u_1 + v_4 \le 11 & u_2 + v_4 \le 20 & u_3 + v_4 \le 18 \end{array}$$

$$u_i, v_j \text{ urs}$$

- we note the simplicity of the dual constraints. We recall that the Row 0 coefficient of the variable x_{ij} is given by the following calculation:

$$\overline{c_{ij}} = \underbrace{\vec{C}_{BV}^{\,T} B^{-1}}_{\vec{y}} \vec{a}_{ij} - c_{ij} \quad \Rightarrow \quad \boxed{\overline{c_{ij}} = u_i + v_j - c_{ij}}$$

$$i, j \quad \begin{bmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

- We can determine the Row 0 coefficient for each variable x_{ij} by computing a corresponding feasible dual solution u_i, v_j. How do we do this? Well, if x_{ij} is in the basis, we know $\overline{c_{ij}} = 0$.

- Consider the following starting solution given by the Northwest Corner Method:

Transportation LP with starting bfs

					Supply
	10 5	2 10	20	11	15
	12	7 5	9 15	20 5	25
	4	14	16	18 10	10
Demand	5	15	15	15	

Basic variables

6 eqns, 7 unknowns

$$\boxed{\begin{array}{lll} x_{11} \rightarrow \bar{C}_{11}=0 \rightarrow & u_1+v_1=c_{11} \\ x_{12} \rightarrow \bar{C}_{12}=0 \rightarrow & u_1+v_2=c_{12} \\ x_{22} \rightarrow \bar{C}_{22}=0 \rightarrow & u_2+v_2=c_{22} \\ x_{23} \rightarrow \bar{C}_{23}=0 \rightarrow & u_2+v_3=c_{23} \\ x_{24} \rightarrow \bar{C}_{24}=0 \rightarrow & u_2+v_4=c_{24} \\ x_{34} \rightarrow \bar{C}_{34}=0 \rightarrow & u_3+v_4=c_{34} \end{array}}$$

- Hence, the dual variables corresponding to this basis satisfy the system

$$\begin{aligned} u_1+v_1&=10 \\ u_1+v_2&=2 \\ u_2+v_2&=7 \\ u_2+v_3&=9 \\ u_2+v_4&=20 \\ u_3+v_4&=18 \end{aligned} \xrightarrow[\text{why? Answer later...}]{\text{Set } u_1=0} \begin{aligned} u_1&=0 \\ v_1&=10 \\ v_2&=2 \\ u_2&=5 \\ v_3&=4 \\ v_4&=15 \\ u_3&=3 \end{aligned} \qquad \begin{aligned} u_1&=k \\ v_1&=10-k \\ v_2&=2-k \\ u_2&=5+k \\ v_3&=4-k \\ v_4&=15-k \\ u_3&=3+k \end{aligned}$$

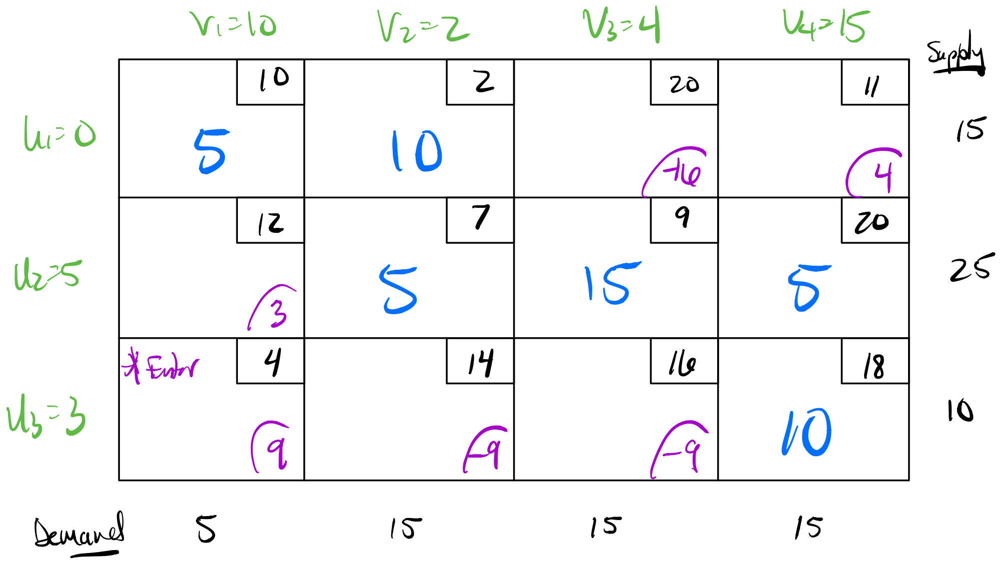
- Using this solution for y, we can find the $\bar{c}_{ij}$ coefficients for the non-basic variables:

$$\left.\begin{array}{ll} x_{13}: & \bar{C}_{13}=u_1+v_3-c_{13}=0+4-20=-16 \\ x_{14}: & \bar{C}_{14}=u_1+v_4-c_{14}=0+15-11=4 \\ x_{21}: & \bar{C}_{21}=u_2+v_1-c_{21}=5+10-12=3 \\ x_{31}: & \bar{C}_{31}=u_3+v_1-c_{31}=3+10-4=9 \leftarrow \text{Enter!} \\ x_{32}: & \bar{C}_{32}=u_3+v_2-c_{32}=3+2-14=-9 \\ x_{33}: & \bar{C}_{33}=u_3+v_3-c_{33}=3+4-16=-9 \end{array}\right\}$$

feasible solution for dual LP.

- On the grid tableau, we can quickly perform these calculations, labeling each row and column with its corresponding dual variable value (starting with $u_1=0$). we can then use the dual variables to calculate the Row 0 coefficients of the non-basic variables.

Transportation LP Calculations

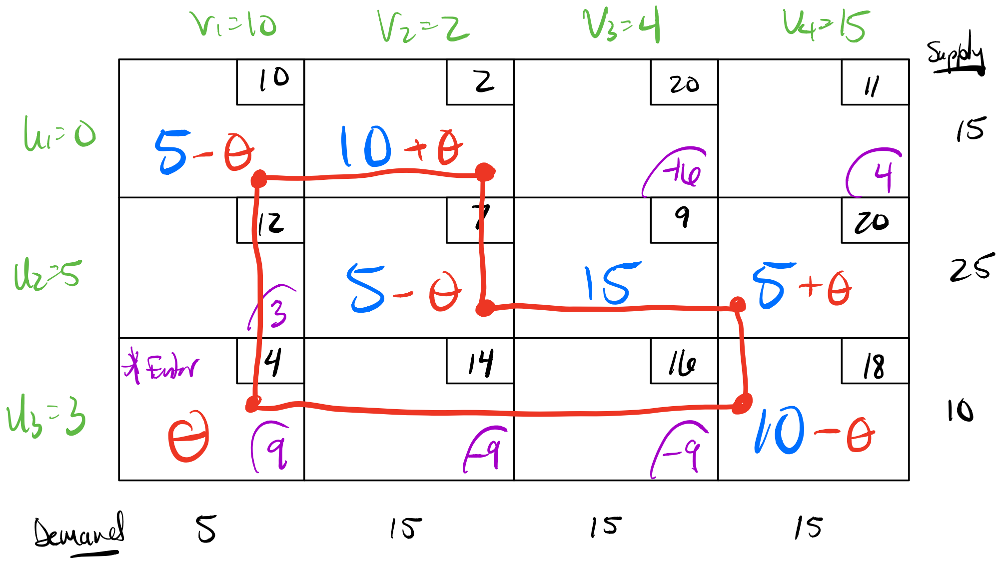


	$v_1=10$	$v_2=2$	$v_3=4$	$v_4=15$	Supply
$u_1=0$	10 5	2 10	20 -16	11 4	15
$u_2=5$	12 3	7 5	9 15	20 5	25
$u_3=3$	*Enter 4 9	14 -9	16 -9	18 10	10
Demand	5	15	15	15	

🟩 – Dual Variables

🟪 – Row 0 Coeffs

- Once we determine the entering variable, we assume it takes on the value θ. If $X_{31}=\theta$, then the variables in Row 3 and Column 1 will have to decrease by θ. Then, other variables within a loop are also affected:



	$v_1=10$	$v_2=2$	$v_3=4$	$v_4=15$	Supply
$u_1=0$	10 $5-\theta$	2 $10+\theta$	20 -16	11 4	15
$u_2=5$	12 3	7 $5-\theta$	9 15	20 $5+\theta$	25
$u_3=3$	*Enter 4 θ 9	14 -9	16 -9	18 $10-\theta$	10
Demand	5	15	15	15	

- We can see to maintain feasibility, we have to satisfy the following inequalities:

x_{11}: $5-\theta \geq 0$

x_{12}: $10+\theta \geq 0$

x_{22}: $5-\theta \geq 0$

x_{24}: $5+\theta \geq 0$

x_{34}: $10-\theta \geq 0$

$\theta = 5 \Rightarrow$ x_{11} or x_{22} must leave the basis.

- Here, we have a tie for the leaving variable. Ties can be broken arbitrarily, but it is best for x_{11} to leave since it has a higher cost coefficient. Hence, we have finished iteration 1:

Transportation LP Completion of 1st Iteration

					Supply
	10	2 15	20	11	15
	12	7 0	9 15	20 10	25
	4 5	14	16	18 5	10
Demand	5	15	15	15	

- We repeat the process until all $\bar{c}_{ij} \leq 0$.

$\boxed{\text{WS}}$ #1 Finish the Transportation Simplex Method

$\boxed{\text{WS}}$ #0 a,b Practice the Transportation Simplex Method

- Now, how do we explain the arbitrary value for u_1? If we're able to let u_1 be anything, this implies that the optimal solution to the primal, say $\vec{x}_{BV}$, corresponds to alternative optima in the dual. This seems to contradict LP theory but it can happen in transportation problems essentially because there will always exist a redundant constraint.

- From the previous example, we must have the following:

$$\underbrace{\begin{aligned} u_1+v_1 &= 10\\ u_1+v_2 &= 2\\ u_2+v_2 &= 7\\ u_2+v_3 &= 9\\ u_2+v_4 &= 20\\ u_3+v_4 &= 18 \end{aligned}}_{\text{6 equations, 7 unknowns underdetermined.}} \Rightarrow 6\left\{\underbrace{\begin{bmatrix} 1&0&0&1&0&0&0\\ 1&0&0&0&1&0&0\\ 0&1&0&0&1&0&0\\ 0&1&0&0&0&1&0\\ 0&1&0&0&0&0&1\\ 0&0&1&0&0&0&1 \end{bmatrix}}_{7}\right. \begin{bmatrix} u_1\\ u_2\\ u_3\\ v_1\\ v_2\\ v_3\\ v_4 \end{bmatrix} = \begin{bmatrix} 10\\ 2\\ 7\\ 9\\ 20\\ 18 \end{bmatrix}$$

This matrix has Rank $=6$, there will be exactly one "free" variable.

- Note, we can let $u_1=k$, k is arbitrary, we get a new set of dual variables that solve this system:

$$\begin{aligned} u_1+v_1 &= 10\\ u_1+v_2 &= 2\\ u_2+v_2 &= 7\\ u_2+v_3 &= 9\\ u_2+v_4 &= 20\\ u_3+v_4 &= 18 \end{aligned} \quad \xrightarrow{\text{Set } u_1=k} \quad \begin{aligned} u_1 &= k\\ v_1 &= 10-k\\ v_2 &= 2-k\\ u_2 &= 5+k\\ v_3 &= 4-k\\ v_4 &= 15-k\\ u_3 &= 3+k \end{aligned}$$

- But the Row 0 coefficients are the same:

$$
\begin{aligned}
x_{13}:&\quad \bar{c}_{13} = u_1+v_3-c_{13} = 0+4-20 = -16\\
x_{14}:&\quad \bar{c}_{14} = u_1+v_4-c_{14} = 0+15-11 = 4\\
x_{21}:&\quad \bar{c}_{21} = u_2+v_1-c_{21} = 5+10-12 = 3\\
x_{31}:&\quad \bar{c}_{31} = u_3+v_1-c_{31} = 3+10-4 = 9\\
x_{32}:&\quad \bar{c}_{32} = u_3+v_2-c_{32} = 3+2-14 = -9\\
x_{33}:&\quad \bar{c}_{33} = u_3+v_3-c_{33} = 3+4-16 = -9
\end{aligned}
$$

- This is due to the fact that one constraint, say the first one, is redundant in the primal LP.

- In general, the balanced transportation LP given by

$$\text{Minimize } z = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij}$$

Subject to

$$\sum_{j=1}^{n} x_{ij} = s_i \quad \text{for } i=1,2,\dots,m$$

$$\sum_{i=1}^{m} x_{ij} = d_j \quad \text{for } j=1,2,\dots,n$$

$$x_{ij} \geq 0 \quad \text{for all } i,j$$

with the associated dual problem given by

$$\text{Maximize } w = \sum_{i=1}^{m} s_i u_i + \sum_{j=1}^{n} d_j v_j$$

Subject to

$$u_i + v_j \leq c_{ij} \quad \text{for } i=1,2,\dots,m, \text{ and } j=1,2,\dots,n$$

$$u_i, v_j \text{ urs}$$

Section 7.4: Sensitivity Analysis for Transportation LPs

- We cover a few techniques for sensitivity analysis of transportation LPs in this section. Since the algorithm for Transportation LPs simplifies, the techniques for sensitivity analysis do as well.

1.) Changing the objective function coefficient for x_{ij}:

$$\text{New } c_{ij} = \text{old } c_{ij} + \Delta c_{ij}$$

- If x_{ij} is a non-basic variable, then

  a.) Compute the new $\bar{c}_{ij}$ for that NBV.

  b.) optimal basis remains the same if new $\bar{c}_{ij} \leq 0$

- If x_{ij} is a basic variable, then

  a.) Compute the new values for dual variables u_i and v_j for this new c_{ij} value.

  b.) Compute new $\bar{c}_{ij}$ values for all NBV's

  c.) Optimal basis is the same if all new $\bar{c}_{ij} \leq 0$.

2.) Changing supply s_i and demand d_j by some amount.

$$\text{New } s_i = \text{Old } s_i + \Delta s_i, \quad \text{New } d_j = \text{Old } d_j + \Delta d_j$$

$$\text{where } \Delta s_i = \Delta d_j$$

- If x_{ij} is a basic variable, then

  a.) simply increase x_{ij} by $\Delta s_i = \Delta d_j$.

  b.) $\text{New } z_{opt} = \text{Old } z_{opt} + u_i \Delta s_i + v_j \Delta d_j$

- If X_{ij} is a non-basic variable, then
  - a.) create a loop involving X_{ij} and some of the basic variables.
  - b.) adjust basic variables in the loop accordingly.

WS #1 a-d work with sensitivity analysis.

Section 7.8: Assignment Problems

- Assignment Problems are a specific type of Transportation LP where the supply and demand values are all equal to 1.
- Although the structure is simplified, the algorithm we employ for Transportation LPs can be very inefficient.

- Assignment Problem: Suppose we wish to assign n workers to n jobs, then defined

$$X_{ij} = \begin{cases} 1 & \text{if worker } i \text{ is assigned job } j \\ 0 & \text{otherwise.} \end{cases}$$

Then the LP takes the form:

$$\text{Minimize} \quad z = \sum_{i=1}^{n} \sum_{j=1}^{n} C_{ij} X_{ij}$$

Subject to

$$\sum_{i=1}^{n} X_{ij} = 1 \quad \text{for } j = 1, 2, \ldots, n$$

$$\sum_{j=1}^{n} X_{ij} = 1 \quad \text{for } i = 1, 2, \ldots, n$$

where C_{ij} is the cost of assigning worker i to job j.

- In any Transportation LP, if the supply/demand values are integers then our solution will be integer-valued. Thus, we can be sure all of our answers are either 0 or 1.

- Because the basic variables are either 0 or 1, a feasible solution may be degenerate which means the simplex algorithm could be vulnerable to cycling. To get around this, we use the Hungarian Method for solving assignment problems by hand.

- Hungarian Method:

1.) Determine p_i, the minimum row cost for every row i for $i=1,2,\ldots,n$. Subtract this value from every element of row i.

2.) Determine q_j, the minimum column cost for every column j for $j=1,2,\ldots,n$. Subtract this value from every element of column j.

3.) From the resulting matrix, attempt to find a feasible assignment among the zero entries.

   3a.) If an assignment can be made, it is optimal.
      $\boxed{\text{COS}}$ #1 work on this scenario.

   3b.) If no feasible assignments can be made,

      i.) Draw the minimum number of vertical and horizontal bars needed to cover zeros.

iii.) Select the smallest uncovered entry, subtract it from every uncovered entry and add it to any zero covered by intersecting lines.

iiii.) If no feasible solution can be made from the resulting zeros, repeat 3a.

$\boxed{\text{WS}}$ #2 work on worst case scenario.

Why are we allowed to subtract diff values from each row and column? Consider the objective function:

<u>Old cost</u>: c_{ij} <u>New cost</u>: $c'_{ij} = c_{ij} - p_i - q_j$

<u>Old z</u>: $z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij}$ <u>New z</u>: $z = \sum_{i=1}^{n} \sum_{j=1}^{n} c'_{ij} x_{ij}$

$$\begin{aligned} z &= \sum_{i=1}^{n} \sum_{j=1}^{n} (c_{ij} - p_i - q_j) x_{ij} \\ &= \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij} - \sum_{i=1}^{n} \sum_{j=1}^{n} p_i x_{ij} - \sum_{i=1}^{n} \sum_{j=1}^{n} q_j x_{ij} \\ &= \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij} - \sum_{i=1}^{n} p_i \underbrace{\sum_{j=1}^{n} x_{ij}}_{=1} - \sum_{j=1}^{n} q_j \underbrace{\sum_{i=1}^{n} x_{ij}}_{=1} \\ &= \text{old } z - \underbrace{\sum_{i=1}^{n} p_i - \sum_{j=1}^{n} q_j}_{\text{Constant not dependent on } X_{ij}} \end{aligned}$$