• Identifying the value of y_i for "≥" or "=" constraints gets ...

but in reality, we need only use the formula

$$\bar{c}_i = \vec{c}_{BV}^{\,T} B^{-1} \vec{a}_i - c_j$$

Consider the following breakdown:

"≤" Constraint : Slack s_i : $\bar{c}_i = \vec{c}_{BV}^{\,T} B^{-1} \vec{a}_i - c_i$

$$\Rightarrow \bar{c}_i = \vec{c}_{BV}^{\,T} B^{-1} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} - 0$$

i^{th} spot.

$$\Rightarrow \bar{c}_i = y_i$$

$$\Rightarrow \boxed{y_i = \bar{c}_i = (\text{Row 0 coeff of } s_i)}$$

"≥" Constraint : Excess e_i : $\bar{c}_i = \vec{c}_{BV}^{\,T} B^{-1} \vec{a}_i - c_i$

(Ignore artificial variable)

$$\Rightarrow \bar{c}_i = \vec{c}_{BV}^{\,T} B^{-1} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ -1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} - 0$$

$$\Rightarrow \bar{c}_i = -y_i$$

$$\Rightarrow \boxed{y_i = -\bar{c}_i = -(\text{Row 0 coeff of } e_i)}$$

"=" Constraint : Artificial a_i : $\bar{c}_i = \vec{c}_{BV}^{\,T} B^{-1} \vec{a}_i - c_i$

$$\Rightarrow \bar{c}_i = \vec{c}_{BV}^{\,T} B^{-1} \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \\ \vdots \\ 0 \end{bmatrix} - \begin{cases} -M & \text{for max LP} \\ M & \text{for min LP} \end{cases}$$

$$\Rightarrow \overline{C_i} = y_i - \begin{cases} -M & \text{for max LP} \\ M & \text{for min LP} \end{cases}$$

$$\Rightarrow y_i = \overline{C_i} + \begin{cases} -M & \text{for max LP} \\ M & \text{for min LP} \end{cases}$$

$$\Rightarrow \boxed{y_i = (\text{Row 0 coefficient of } a_i) + \begin{cases} -M & \text{for max LP} \\ M & \text{for min LP} \end{cases}}$$

$\boxed{\text{WS}}$ work on constructing the dual solution.

Section 6.8: Shadow Prices (and Feasibility Ranges)

- Recall the definition of a shadow price from earlier in this chapter:
- Def: The shadow price of the i^{th} constraint is the amount by which the optimal z-value is improved (increased in a max problem, decreased in a min problem) if we increased b_i by 1 unit (from b_i to b_i+1).
- Note: the definition above assumes $\vec{X}_{BV}$ stays optimal.
- We can use the Dual Theorem to show that the solution to the dual problem, $\vec{y} = [y_1 \dots y_i \dots y_m]$, contains the shadow price for each constraint. Consider the following:

$$\underset{\text{Primal}}{z_{opt} = \sum_{j=1}^{n} c_j x_j} = \underset{\text{Dual}}{\sum_{i=1}^{m} b_i y_i = w_{opt}}$$

- Assume we change the right hand side of the i^{th} constraint to be $b_i + \Delta b_i$ where Δb_i is the deviation in the right-hand side. Also, assume Δb_i is within the feasibility range so that the current basis remains optimal. Then we have

$$\underbrace{[\text{New } z_{opt}] = \sum_{j=1}^{n} c_j x_j}_{\text{Primal}} = \underbrace{\sum_{i=1}^{m} (b_i + \Delta b_i) y_i}_{\text{Dual}}$$

$$= \sum_{i=1}^{m} b_i y_i + \sum_{i=1}^{m} \Delta b_i y_i$$

$$= [\text{Old } z_{opt}] + \underbrace{\sum_{i=1}^{m} \Delta b_i y_i}_{\text{Sum of all shadow prices applied to deviations.}}$$

- Note, for a max LP with only "$\le$" constraints, if $\Delta b_i > 0$ then the z_{opt} value will only increase since $y_i \ge 0$ for all i. This means, if we add resources to a limited resource constraint, our optimal value (revenue) should increase.

- On the contrary, for "$\ge$" or "$=$" constraints, y_i may be negative and so if $\Delta b_i > 0$, we may lose revenue. This corresponds to a demand constraint — required production may be costly depending on y_i.

#1 Working with interpreting shadow prices.

- We can also use the dual theorem to easily construct feasibility ranges for changing the right-hand side of a constraint OR changing the right-hand side of all constraints simultaneously. We'll demonstrate with an example:

ToyCo Problem:

$$\max \; z = 3x_1 + 2x_2 + 5x_3$$

S.T.

$$x_1 + 2x_2 + x_3 \le 430 \quad + \Delta b_1$$
$$3x_1 + 2x_3 \le 460 \quad + \Delta b_2$$
$$x_1 + 4x_2 \le 420 \quad + \Delta b_3$$
$$x_1, x_2, x_3 \ge 0$$

Initial tableau

Basic	z	x_1	x_2	x_3	S_1	S_2	S_2	RHS	Δb_1	Δb_2	Δb_3
z	1	-3	-2	-5	0	0	0	0	0	0	0
S_1	0	1	2	3	1	0	0	430	1	0	0
S_2	0	3	0	2	0	1	0	460	0	1	0
S_3	0	1	4	0	0	0	1	420	0	0	1

Same!

Optimal Tableau

Basic	z	x_1	x_2	x_3	S_1	S_2	S_2	RHS	Δb_1	Δb_2	Δb_3
z	1	4	0	0	1	2	0	1350	1	2	0
x_2	0	-1/4	1	0	1/2	-1/4	0	100	1/2	-1/4	0
x_3	0	3/2	0	1	0	1/2	0	230	0	1/2	0
S_3	0	2	0	0	-2	1	1	20	-2	1	1

B^{-1} $\qquad$ B^{-1}

- We can quickly read from the optimal tableau that the objective function at optimality is

$$Z_{opt} = 1350 + \Delta b_1 + 2\Delta b_2$$

which confirms our previous result.

- Also, we can quickly read from the table how the optimal basis values are changed:

$$x_2 = 100 + \tfrac{1}{2}\Delta b_1 - \tfrac{1}{4}\Delta b_2$$

$$x_3 = 230 \qquad + \tfrac{1}{2}\Delta b_2$$

$$S_3 = 20 - 2\Delta b_1 + \Delta b_2 + 1\Delta b_3$$

We see that the elements of B^{-1} are sorta like the "shadow prices" for the actual optimal basis values.

- Note that in order for x_2, x_3, S_3 to remain optimal, we must have $x_2 \geq 0$, $x_3 \geq 0$, $S_3 \geq 0$:

$$100 + \tfrac{1}{2}\Delta b_1 - \tfrac{1}{4}\Delta b_2 \geq 0 \Rightarrow -\tfrac{1}{2}\Delta b_1 + \tfrac{1}{4}\Delta b_2 \leq 100$$

$$230 + \tfrac{1}{2}\Delta b_2 \geq 0 \Rightarrow -\tfrac{1}{2}\Delta b_2 \leq 230$$

$$20 - 2\Delta b_1 + \Delta b_2 + \Delta b_3 \geq 0 \Rightarrow 2\Delta b_1 - \Delta b_2 - \Delta b_3 \leq 20$$

This defines a region in 3-Dim space $(\Delta b_1, \Delta b_2, \Delta b_3)$ — "simultaneous" feasibility range.

Section 6.9: Duality & Sensitivity Analysis.

- In the proof of the Dual Theorem, we established the following fact:

  "Suppose BV is a feasible set of variables for the primal LP. Then BV is optimal if and only if $\vec{c}_{BV}^{\,T} B^{-1}$ (the dual solution) is feasible."

- We can use this double-edged sword to conduct sensitivity analysis by simply maintaining feasibility in the dual problem.

1.) Changing an objective function coefficient:

Variable altered: c_j for some j

Let c_j be the "new" value the obj. fun coeff. Then BV is still optimal if the corresponding dual constraint

$$\sum_{i=1}^{m} a_{ij} y_i \quad \begin{matrix} (\geq) \\ (\leq) \\ (=) \end{matrix} \quad c_j$$

is still satisfied for the optimal $\bar{y}$ values.

2.) Changing the column of a non-basic variable:

Variables altered: a_{ij} for $i = 1, 2, \ldots, m$

BV is still optimal if the corresponding dual constraint

$$\sum_{i=1}^{m} a_{ij} y_i \quad \begin{matrix}(\geq)\\(\leq)\\(=)\end{matrix} \quad c_j$$

is still satisfied for the optimal $\vec{y}$ values.

3.) **Adding a new activity:**

Variables added: $a_{i,n+1}$ for $i = 1, 2, \dots m$ and c_{n+1}

BV is still optimal if the corresponding dual constraint

$$\sum_{i=1}^{m} a_{i,n+1} y_i \quad \begin{matrix}(\geq)\\(\leq)\\(=)\end{matrix} \quad c_{n+1}$$

is still satisfied for the optimal $\vec{y}$ values.

$\boxed{\text{WS}}$ #1 Walk with an example to show how sensitivity analysis can be done using the dual constraints.

Section 6.10: Complementary Slackness

- One last idea that we can exploit from the relationship of the primal and dual problems is the idea of complementary slackness. Here, we formalize the relationship between optimal solutions of each problem and the values of the primal slack variables and the dual excess variables.
- Note: we'll consider the normal primal max LP and its dual.

Theorem: (complementary slackness) let $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ be a primal feasible solution and $\vec{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix}$ be a dual feasible solution. Then $\vec{x}$ is primal optimal and $\vec{y}$ is dual optimal if and only if

$$s_i y_i = 0 \quad \text{for } i = 1, 2, \dots, m$$

$$e_j x_j = 0 \quad \text{for } j = 1, 2, \dots, n$$

where s_i is the i^{th} slack variable for the primal LP and e_j is the j^{th} excess variable for the dual LP.

Note: The forward conclusion of this theorem means that the following is true:

$$\left.\begin{array}{lll} (1) \text{ if } s_i > 0 & \text{then} & y_i = 0 \\ (2) \text{ if } y_i > 0 & \text{then} & s_i = 0 \end{array}\right\} \text{i.e. } s_i \text{ or } y_i \text{ must be zero for all } i.$$

$$\left.\begin{array}{lll} (3) \text{ if } e_j > 0 & \text{then} & x_j = 0 \\ (4) \text{ if } x_j > 0 & \text{then} & e_j = 0 \end{array}\right\} \text{i.e. } e_j \text{ or } x_j \text{ must be zero for all } j.$$

Proof: Suppose $\vec{x}$ and $\vec{y}$ are both feasible solutions to
($\Rightarrow$) their respective LPs given below

Primal	Dual
Max $z = \vec{c}^T \vec{x}$	Min $w = \vec{y}^T \vec{b}$
S.T. $A\vec{x} \le \vec{b}$	S.T. $A^T \vec{y} \ge \vec{c}$
$\vec{x} \ge \vec{0}$	$\vec{y} \ge \vec{0}$

By weak duality, we know the following holds

$$\vec{c}^T\vec{x} \le \vec{y}^T A\vec{x} \le \vec{y}^T\vec{b} \qquad (1)$$

since $\vec{x}$ and $\vec{y}$ are feasible and $A^T\vec{y} \ge \vec{c}$ is equivalent to $\vec{c}^T \ge \vec{y}^T A$. Define the following slack vector, $\vec{s}$, and excess vector $\vec{e}$:

$$(n \times 1) \quad \begin{aligned} \vec{s} &= \vec{b} - A\vec{x} \\ \vec{s} &\ge \vec{0} \end{aligned} \qquad\qquad \begin{aligned} \vec{e} &= A^T\vec{y} - \vec{c} \\ \vec{e} &\ge \vec{0} \end{aligned} \quad (m \times 1)$$

Now, suppose $\vec{x}$ and $\vec{y}$ are optimal, then Equation (1) becomes equality by the Dual Theorem:

$$z_{opt} = c^T\vec{x} = \vec{y}^T A\vec{x} = \vec{y}^T\vec{b} = w_{opt}$$

This equality implies two separate equations:

$$\vec{c}^T\vec{x} = \vec{y}^T A\vec{x} \qquad\qquad \vec{y}^T A\vec{x} = \vec{y}^T\vec{b}$$

$$\Rightarrow \vec{x}^T\vec{c} = \vec{x}^T A^T\vec{y}$$

$$\Rightarrow \vec{x}^T A^T\vec{y} - \vec{x}^T\vec{c} = 0 \qquad\qquad \Rightarrow \vec{y}^T\vec{b} - \vec{y}^T A\vec{x} = 0$$

$$\Rightarrow \vec{x}^T(A^T\vec{y} - \vec{c}) = 0 \qquad\qquad \Rightarrow \vec{y}^T(\vec{b} - A\vec{x}) = 0$$

$$\Rightarrow \vec{x}^T\vec{e} = 0 \qquad\qquad \Rightarrow \vec{y}^T\vec{s} = 0$$

$$\Rightarrow \sum_{j=1}^{n} e_j x_j = 0 \qquad\qquad \Rightarrow \sum_{i=1}^{n} s_i y_i = 0$$

Here, have two sums, each containing non-negative terms, that sum to zero. This implies that every term of each sum must be zero, i.e.

$$e_j x_j = 0 \quad \text{for } j = 1, 2, \ldots, n$$

$$s_i y_i = 0 \quad \text{for } i = 1, 2, \ldots, m$$

as desired.

($\Leftarrow$) Assume $\vec{x}$ and $\vec{y}$ are feasible solutions to their respective LP problems. Assume

$$e_j x_j = 0 \quad \text{for } j = 1, 2, \ldots, n$$

$$s_i y_i = 0 \quad \text{for } i = 1, 2, \ldots, m$$

then

$$\vec{x}^T \vec{e} = 0 \qquad \qquad \vec{y}^T \vec{s} = 0$$

Working backwards as in the previous proof, we have

$$c^T \vec{x} = \vec{y}^T A \vec{x} = \vec{y}^T \vec{b}$$

Since $\vec{c}^T \vec{x} = \vec{y}^T \vec{b}$, we know $\vec{x}$ and $\vec{y}$ are optimal by Lemma 2. ∎

<u>Main Result</u>: If a primal or dual constraint is non-binding ($e_i > 0$ or $s_i > 0$), then the corresponding variable in the complementary problem is 0.

- The result that $s_i = 0$ implies $y_i > 0$ and $y_i = 0$ implies $s_i > 0$ makes sense mechanistically, since $s_i = 0$ implies the Row 0 value of s_i, (i.e. y_i), is non-negative. But economically does it make sense?

  "If $s_i > 0$, the resource is abundant (some was unused) in primal problem, thus its shadow price is 0 — an extra unit of that resource is worthless."

- The result that $e_j = 0$ implies $x_j > 0$ and $x_j = 0$ implies $e_j > 0$ means the following, economically:

$$e_j = \underbrace{\sum_{i=1}^{n} a_{ij} y_i}_{\text{value of resources used to make product } j} - \underbrace{c_j}_{\text{sales price.}}$$

If $e_j > 0$, we're not selling the product x_j at a high enough price and so we shouldn't make any! $\Rightarrow x_j = 0$.

"For any x_j in the primal basis, the marginal revenue must match the marginal production cost i.e. $e_j = 0$."

$\boxed{\text{WS}}$ #1 Practical use of complementary slackness.