MAT 362 - Operations Research II

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CHAPTER 6 - Sensitivity Analysis & Duality

Section 6.5: Finding the Dual of an LP.

- For any LP, there is a dual problem. The original LP problem, typically called the primal LP, has a specific relationship with the dual LP. This relationship will provide a deeper understanding of sensitivity analysis. Also, the theory is nice!
- First, for any normal max or min LP, the dual is not too difficult to formulate. Consider the following, written in matrix vector notation:

Primal Max LP (normal)		Dual Min LP (normal)
Max $z = \vec{c}^T\vec{x}$	$\Longleftrightarrow$	Min $w = \vec{b}^T\vec{y}$
S.T. $A\vec{x} \leq \vec{b}$		S.T. $A^T\vec{y} \geq \vec{c}$
$\vec{x} \geq \vec{0}$		$\vec{y} \geq \vec{0}$

Decision Variables: $\vec{x} = n \times 1$ with m constraints

Constraints / Obj. Value Coeff: $A = (m \times n)$, $\vec{c} = n \times 1$, $\vec{b} = m \times 1$

Decision Variables: $\vec{y} = m \times 1$ with n constraints.

- If the primal LP is a min problem, the dual LP is a max problem and the same transformation occurs. The primal LP is always in the variable $\vec{x}$ with A being an $m \times n$ matrix. This is different from the text. ($\vec{y}$ is always the dual variable)

$\boxed{\text{WS}}$ #1, #2 work on finding the dual of a normal primal LP.

- If the primal LP is not in normal form, then a technique can be applied to arrive at the dual problem. We'll demonstrate by example, then summarize the results.

$\boxed{\text{WS}}$ #3 Find the dual of the non-normal LP.

- Rules for constructing the dual problem:

Max LP		Min LP
Constraints		Variables
$\geq$	$\longleftrightarrow$	≤ 0
$\leq$	$\longleftrightarrow$	≥ 0
$=$	$\longleftrightarrow$	urs
Variables		Constraints
≥ 0	$\longleftrightarrow$	$\geq$
≤ 0	$\longleftrightarrow$	$\leq$
urs	$\longleftrightarrow$	$=$

Section 6.6: Economic Interpretation of the Dual

- If we consider the case of a normal Max LP (Primal) as a resource allocation model (i.e. maximizing revenue/profit given limited resources) then the interpretation of the normal Min LP (dual) becomes more clear. We can use a dimensional analysis to deduce the units of the dual variables. Later, we can also interpret the reason why the optimal solution to the primal LP is identical to the optimal solution to the dual LP (i.e. $z = w$ at optimality).

- Consider the following general interpretations:

$$\boxed{\begin{gathered} A = [a_{ij}] \in \mathbb{R}^{m \times n} \\ i = 1,2,\dots,m \\ j = 1,2,\dots,n \end{gathered}}$$

PRIMAL

- Resource Allocation Model
- Obj. Fun.:

$$\text{Max } z = \sum_{j=1}^{n} c_j x_j$$

$[z]$ = Revenue (\$)

$[x_j]$ = unit of product j

$[c_j]$ = Revenue (\$)/unit of product j

- Constraints:

$$\sum_{j=1}^{n} a_{ij} x_j \le b_i, \quad i = 1,2,\dots,m$$

$$x_j \ge 0, \quad j = 1,2,\dots,n$$

DUAL

- Worth Assessment Model
- Obj. Fun.:

$$\text{Min } w = \sum_{i=1}^{m} b_i y_i$$

$[w]$ = Total worth (\$)

$[y_i]$ = Price (\$)/unit of resource i

$[b_i]$ = units of resource i

- Constraints:

$$\sum_{i=1}^{m} a_{ij} y_i \le c_j, \quad j = 1,2,\dots,n$$

$$y_i \ge 0, \quad i = 1,2,\dots,m$$

(4)

$[b_i]$ = units of resource i	$[c_j]$ = Price (\$)/unit of product j
$[a_{ij}]$ = units of resource i / unit of product j	$[a_{ij}]$ = units of resource i / unit of product j
	$[y_i]$ = Price (\$)/unit of resource i (SHADOW PRICE or DUAL PRICE)
Goal: Maximize revenue given limited resources.	**Goal**: Minimize the overall cost of the resources based on what revenue the products made from them can generate.

- From this dimensional analysis, we see that y_i is the worth per one unit of resource i. We call this the <u>shadow</u> or <u>dual</u> price.
- Therefore, given m resources, each of quantity b_i, the objective function $w = \sum_{i=1}^{m} b_i y_i$ is the total worth of all of the resources. The smaller the value of w, the more we are exploiting our resources.
- Consider the two objective functions. For any feasible (not optimal) solutions to the primal and the dual problems, we must have

$$z < w$$

Revenue: $z = \sum_{j=1}^{n} c_j x_j$

Worth: $w = \sum_{i=1}^{m} b_i y_i$

- Optimality is achieved when the revenue is equal to the worth of the resources i.e. resources have be exploited.

Section 6.7: The Dual Theorem and its Consequences

- In this section, we prove the Dual Theorem for a normal max primal LP which shows that the primal and dual problems have the same optimal objective function value (assuming they both have optimal solutions.

- First we recap a few things:

  1.) For any matrix or vector

$$(A^T)^T = A \quad \text{and} \quad (\vec{x}^T)^T = \vec{x}$$

  2.) For any matrix-matrix product AB or matrix vector product $A\vec{x}$, we have

$$(AB)^T = B^T A^T \quad \text{and} \quad (A\vec{x})^T = \vec{x}^T A^T$$

  3.) For any two vectors $\vec{x}, \vec{y}$ of the same size we have

$$\vec{x}^T\vec{y} \in \mathbb{R} \implies \vec{x}^T\vec{y} = (\vec{x}^T\vec{y})^T = \vec{y}^T\vec{x}.$$

- In the lemma and theorem to follow, we consider the normal max LP for the primal and the normal min LP for the dual:

Primal:

$$\text{Max } z = \vec{c}^T\vec{x}$$
$$\text{S.T.} \quad A\vec{x} \le \vec{b}$$
$$\vec{x} \ge \vec{0}$$

Dual:

$$\text{Min } w = \vec{b}^T\vec{y}$$
$$\text{S.T.} \quad A^T\vec{y} \ge \vec{c}$$
$$\vec{y} \ge \vec{0}$$

- Now, we prove a Lemma called weak Duality.

- **Lemma 1:** (Weak Duality) Let $\vec{x} \in \mathbb{R}^{n \times 1}$ be any feasible solution to the normal primal max LP and let $\vec{y} \in \mathbb{R}^{m \times 1}$ be a feasible solution to the normal dual min LP. Then $z \leq w$, where z is the objective function value for the primal and w is the objective function value for the dual.

Proof: Let $\vec{x}$ and $\vec{y}$ be feasible solutions, then we know $\vec{x} \geq \vec{0}$ and $\vec{y} \geq \vec{0}$, and

$$A\vec{x} \leq \vec{b} \quad \text{and} \quad A^T\vec{y} \geq \vec{c}.$$

It follows that

$$(A^T\vec{y})^T \geq (\vec{c})^T \Rightarrow \boxed{\vec{y}^T A \geq \vec{c}^T}$$

Now, consider the objective function value z:

$$z = \vec{c}^T\vec{x} \leq (\vec{y}^T A)\vec{x} = \vec{y}^T(A\vec{x}) \leq \vec{y}^T\vec{b} = \vec{b}^T\vec{y} = w,$$

as desired. ∎

- The following Lemma is a consequence of weak Duality.

- **Lemma 2:** Let $\hat{x}$ be a feasible solution to the primal and $\hat{y}$ be a feasible solution to the dual such that $\vec{c}^T\hat{x} = \vec{b}^T\hat{y}$. Then $\hat{x}$ is optimal for the primal and $\hat{y}$ is optimal for the dual.

- Proof: Suppose $\vec{x}$ is a feasible solution, then from Weak Duality we know

$$\vec{c}^T \vec{x} \leq \vec{b}^T \hat{y}$$

since $\hat{y}$ is a feasible solution. Therefore, $\vec{b}^T\hat{y}$ is an upper bound for any feasible solution to the primal and so the optimal solution cannot be greater than that value. But we know $\vec{c}^T\hat{x} = \vec{b}^T\hat{y}$; thus $\hat{x}$ is optimal.

Likewise, suppose $\vec{y}$ is any feasible solution to the dual LP. Then

$$\vec{c}^T\hat{x} \leq \vec{b}^T\vec{y}$$

is a lower bound for the optimal solution of the dual. But $\vec{b}^T\hat{y} = \vec{c}^T\hat{x}$, which means $\hat{y}$ is optimal. ∎

- Now, before we get started on the Dual Theorem, we recall a few things. Suppose BV is a set of basic variables that is the optimal solution, then

1.) $\vec{C}_{BV}$ is $m \times 1$ and B^{-1} is $m \times m$, where m is the number of constraints.

2.) The value $\bar{c}_j = \vec{c}_{BV}^T B^{-1} \bar{a}_j - c_j$ is the Row 0 coefficient corresponding to the variable x_j.

3.) Suppose s_i was a slack variable to the max LP. Then the row 0 coefficient is the i^{th} value of the row vector

$$\vec{c}_{BV}^T B^{-1}$$

Since

$$\bar{c}_i = \vec{c}_{BV}^{\,T} B^{-1} a_i - c_i$$

$$\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \leftarrow i^{th} \text{ element} \qquad = 0$$

$$\Rightarrow \quad \bar{c}_i = i^{th} \text{ element of } \vec{c}_{BV}^{\,T} B^{-1}$$

- Keeping all of this in mind, we can prove the Duality Theorem.

- **Theorem**: (Duality Theorem) Suppose BV is an optimal basis for the primal. Then $\vec{c}_{BV}^{\,T} B^{-1}$ is an optimal solution to the dual.

- **Proof**: we first add slack variables to the primal max LP:

$$\begin{array}{ll}
\text{Max} & z = c_1 x_1 + \cdots + c_n x_n \\
\text{S.T.} & a_{11} x_1 + \cdots + a_{1n} x_n + s_1 = b_1 \\
& \vdots \qquad\qquad\qquad\qquad s_i \qquad\quad = b_i \\
& a_{m1} x_1 + \cdots + a_{mn} x_n + \cdots + s_m = b_m
\end{array}$$

Suppose BV is an optimal basis, and define

$$\vec{c}_{BV}^{\,T} B^{-1} = \vec{y} = [y_1 \;\; y_2 \;\; \cdots \;\; y_m]$$

to be a $1 \times m$ row vector. Here, we see that y_i is the i^{th} element of $\vec{c}_{BV}^{\,T} B^{-1}$. Since BV is optimal, we know the Row 0 coefficient for each element x_j is non-negative (note: the x_j's in BV have 0 in Row 0). These coefficients are given by

Row 0 cost ≥ 0

$$\bar{C}_j = \vec{C}_{BV}^T B^{-1} \vec{a}_j - c_j \quad \text{for } j = 1, 2, \ldots, n$$

$$\Rightarrow \quad \boxed{C_{BV}^T B^{-1} \vec{a}_j - c_j \geq 0}$$

$$\Rightarrow \quad [y_1 \cdots y_i \cdots y_m] \begin{bmatrix} a_{1j} \\ \vdots \\ a_{ij} \\ \vdots \\ a_{mj} \end{bmatrix} - c_j \geq 0$$

$$\Rightarrow \quad a_{1j} y_1 + \cdots + a_{ij} y_i + \cdots + a_{mj} y_m - c_j \geq 0$$

$$\Rightarrow \quad \boxed{\sum_{i=1}^{m} a_{ij} y_i \geq c_j} \quad \text{for } j = 1, 2, \ldots, n$$

Thus, $\vec{C}_{BV}^T B^{-1} = \vec{y}$ satisfies the n dual constraints!

Now, consider the slack variables, s_i, for which there are m. They too have non-negative values in Row 0. Thus,

$$\bar{C}_i = C_{BV}^T B^{-1} \vec{a}_i - C_i \geq 0$$

$$\Rightarrow \quad [y_1 \cdots y_i \cdots y_m] \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} - 0 \geq 0$$

i^{th} element

$$\Rightarrow \quad \boxed{y_i \geq 0} \quad \text{for } i = 1, 2, \ldots, m$$

Thus, $\vec{C}_{BV}^T B^{-1} = \vec{y}$ consists of all non-negative values. Therefore, $\vec{y}$ is a <u>feasible</u> solution to the dual problem!

Finally, we know the objective value of the primal is

$$
\begin{aligned}
z_{opt} &= \vec{c}_{BV}^{\,T} B^{-1} \vec{b} \\
&= \begin{bmatrix} y_1 & \cdots & y_m \end{bmatrix} \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix} \\
&= b_1 y_1 + \cdots + b_m y_m \\
&= w
\end{aligned}
$$

Therefore, the dual objective value for dual evaluated at $\vec{c}_{BV}^{\,T} B^{-1}$ is equal to the optimal primal objective value. By Lemma 2, it follows that $\vec{c}_{BV}^{\,T} B^{-1}$ is optimal for the dual LP and $z_{opt} = w_{opt}$. ∎

WS Prove Lemma 3 using Weak Duality.

Take away: In the max primal LP, we initially add in m slack variables. The coefficient in Row 0 of the optimal tableau of slack variable s_i is exactly the i^{th} element of $\vec{c}_{BV}^{\,T} B^{-1}$. Thus,

$$y_i = (\text{Row } 0 \text{ coeff of } s_i \text{ at optimality})$$

for $i = 1, 2, \ldots, m$.

WS Using primal optimal solutions to deduce dual solutions.