MAT 181 - Calculus I

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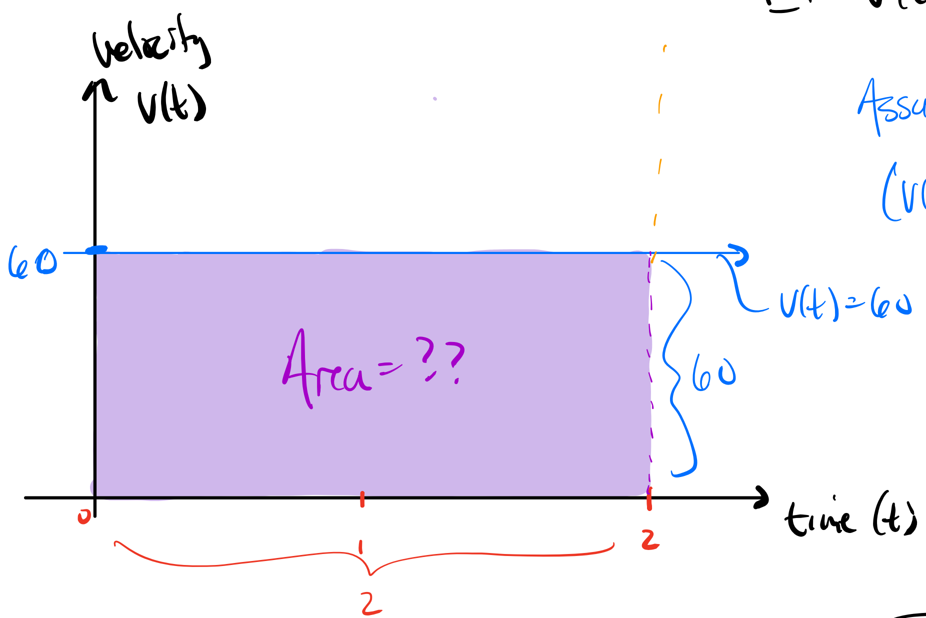
CHAPTER 5 - Integration

Section 5.1: Areas, Distances, and Riemann Sums.

- Recall, Distance = Rate (velocity) × Time. If consider the area "under" a velocity curve, this value is the total distance traveled.

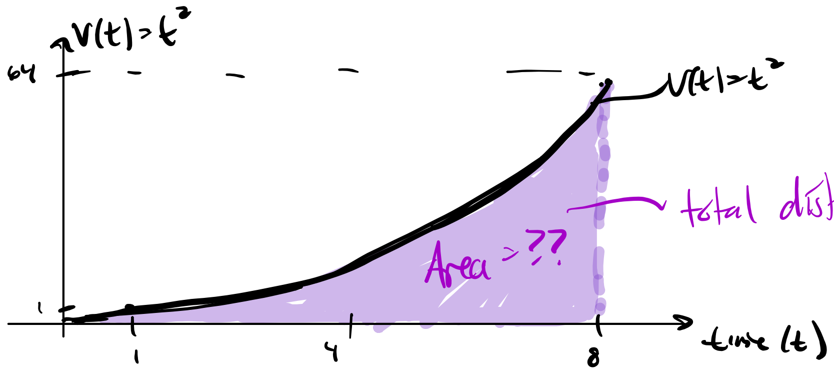
Ex: $V(t)$ = velocity at time t.

Assume $V(t) = 60 \text{ miles/hr}$ ($V(t)$ is constant)



$$\text{Area} = l \cdot w = (2 \cancel{\text{hours}}) \times \left(60 \frac{\text{miles}}{\cancel{\text{hr}}}\right) = \underbrace{\boxed{120 \text{ miles}}}_{\text{total dist. traveled}}$$

Ex: Suppose $V(t) = t^2$. What is the total dist. traveled after 8 hours?



a bunch of subdivisions

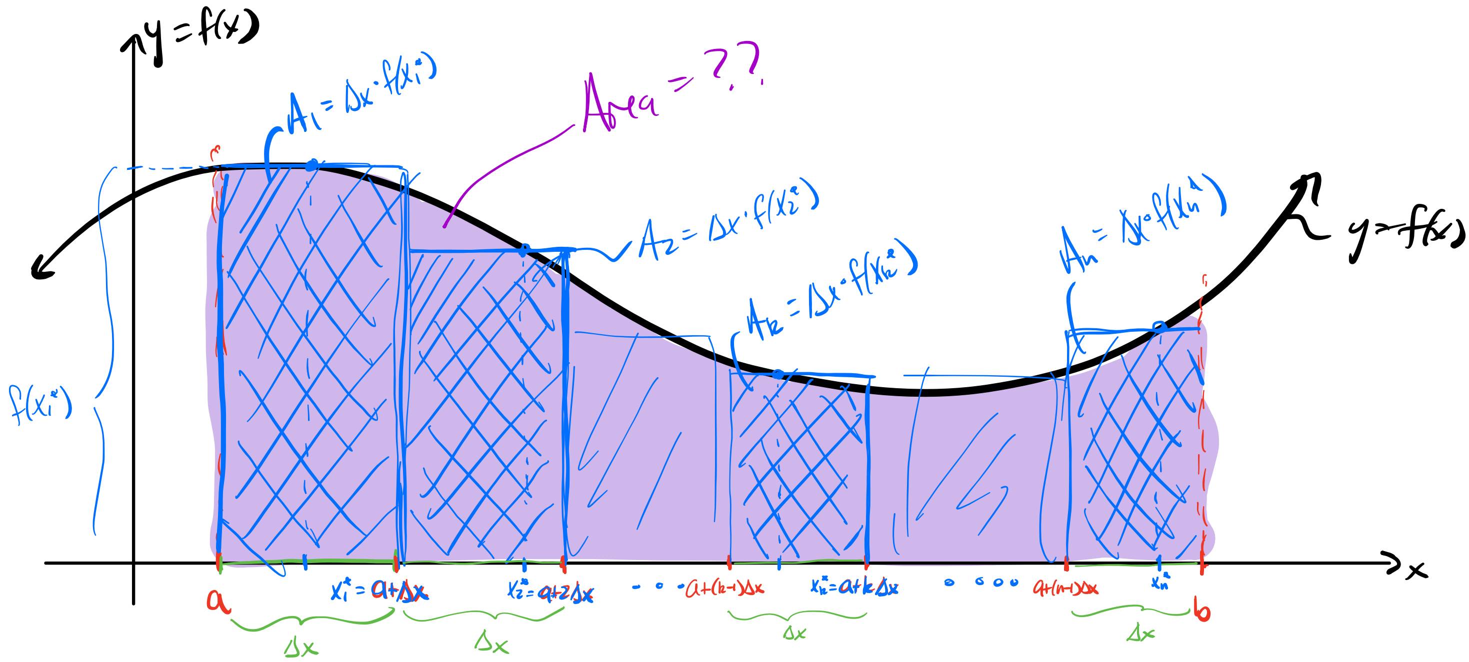
Approximation: Create a partition of the interval [0,8], then construct Riemann Sum Rectangles

- Consider the function $f(x)$ on the interval $[a,b]$. Then the Riemann Sum approximation to the area bounded by the curve $f(x)$ and the x-axis is given by the following

$$[\text{Area}] \approx \Delta x \cdot f(x_1^*) + \Delta x \cdot f(x_2^*) + \cdots + \Delta x \cdot f(x_n^*)$$

where n is the # of subintervals, $\Delta x = \frac{b-a}{n}$ is the width of each subinterval, and x_k^* is a sample point in the k^{th} sub-interval.

k is the "index" which ranges from 1 to n.



width: $\Delta x = \frac{b-a}{n}$

Riemann Sum: $\Delta x \cdot f(x_1^*) + \Delta x \cdot f(x_2^*) + \cdots + \Delta x \cdot f(x_k^*) + \cdots + \Delta x \cdot f(x_n^*)$

• **Summation notation:** The Riemann Sum can be written compactly as

Index upper bound

$$f(x_1^*)\cdot\Delta x + f(x_2^*)\cdot\Delta x + \cdots + f(x_n^*)\,\Delta x = \sum_{k=1}^{n} \underbrace{f(x_k^*)\cdot\Delta x}_{\text{Summand}}$$

Index lower bound

• **Summation Identities:**

1.) $$\sum_{k=1}^{n} 1 = n = \underbrace{1+1+1+\cdots+1}_{n \text{ times}}$$

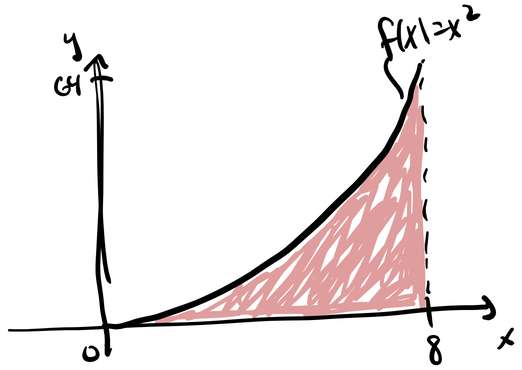
2.) $$\sum_{k=1}^{n} k = \frac{n(n+1)}{2} = 1+2+3+\cdots+(n-2)+(n-1)+n$$

n+1
n+1
n+1

3.) $$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} = \frac{n^3}{3}+\frac{n^2}{2}+\frac{n}{6}$$

4.) $$\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2 = \frac{n^4}{4}+\frac{n^3}{2}+\frac{n^2}{4}$$

• So far we've computed 3 Riemann Sums to approximate the area under $f(x)=x^2$ on the interval $[0,8]$:



From worksheet:

$n=2$, midpoints: [Area] $\approx$ 160

$n=4$, midpoints: [Area] $\approx$ 168

$n=8$, midpoints: [Area] $\approx$ 170

$\vdots$ $\qquad\qquad$ $\vdots$

As $n\to\infty$, [Area] $\longrightarrow$???

It will definitely be the true area.

General, Right-Hand Riemann Sum:

Consider a function $f(x)$ on the interval $[a,b]$. Let n be the number of rectangles. Then $\Delta x = \frac{b-a}{n}$ is the width of the subintervals. We use the right endpoint of each subinterval as the sample point, given by

$$x_k^* = a + k\Delta x, \quad \text{for } k = 1, 2, \ldots, n.$$

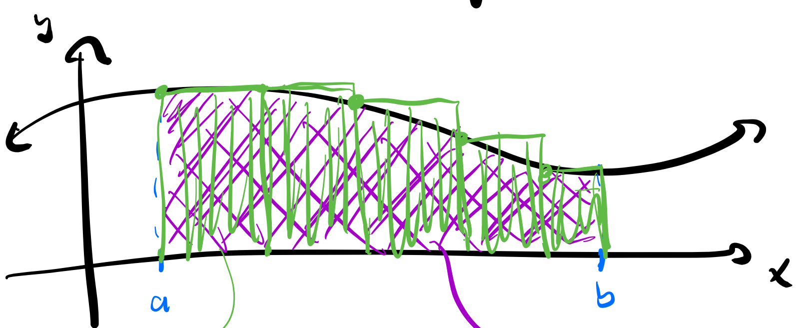
The Right-Hand Riemann Sum with n subintervals is

$$R_n = \sum_{k=1}^{n} f(x_k^*)\Delta x.$$

The area under $f(x)$ on $[a,b]$ is approximated by R_n.

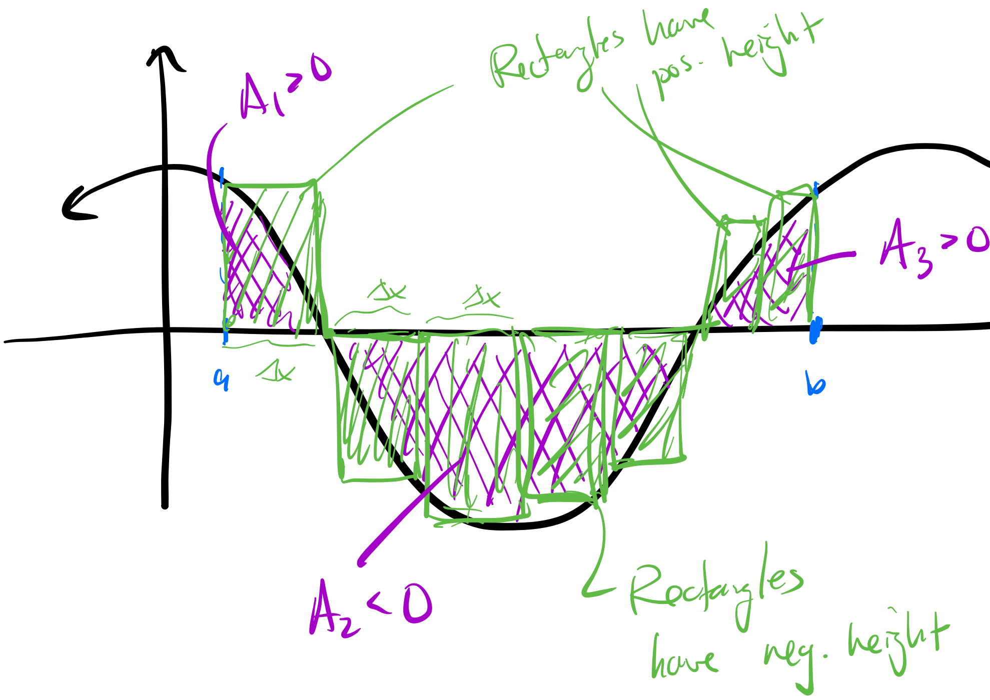
Section 5.2: The Definite Integral

Recap: Area "under" the curve $y=f(x)$ on $[a,b]$.



$$\underbrace{\sum_{k=1}^{n} f(x_k^*)\Delta x}_{\text{Riemann Sum}} \approx \text{True Area}$$

This area "under" the curve could be negative!



Thus, the area "under" the curve is a <u>signed area</u> (net area) bounded by the curve and the x-axis.

<u>Def</u>: The signed area bounded by $y = f(x)$ and the x-axis on the interval $[a,b]$ is called the <u>definite integral of $f(x)$ from a to b</u>, denoted by

$$\int_a^b f(x)\,dx = \text{an area (it's a real number!)}$$

Break-down:

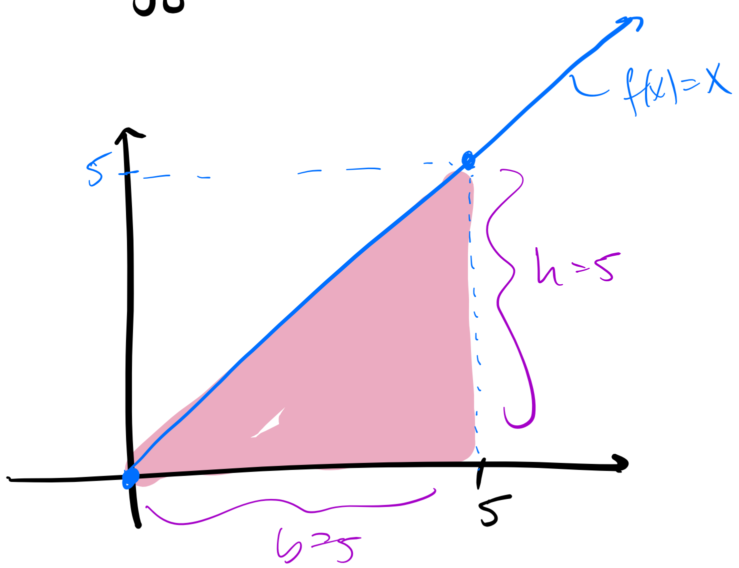
upper bound

Integral symbol

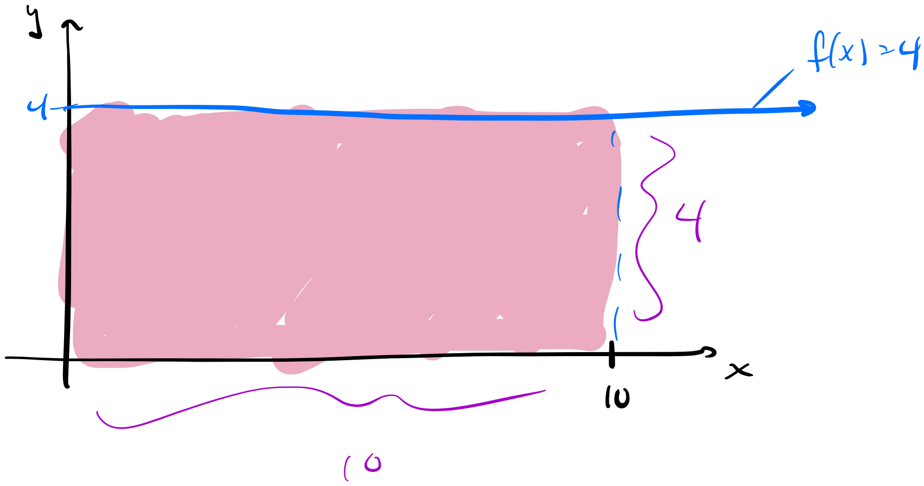
$$\int_a^b \underbrace{f(x)}_{\text{integrand}} \, \underbrace{dx}_{\text{differential element}}$$

lower bound

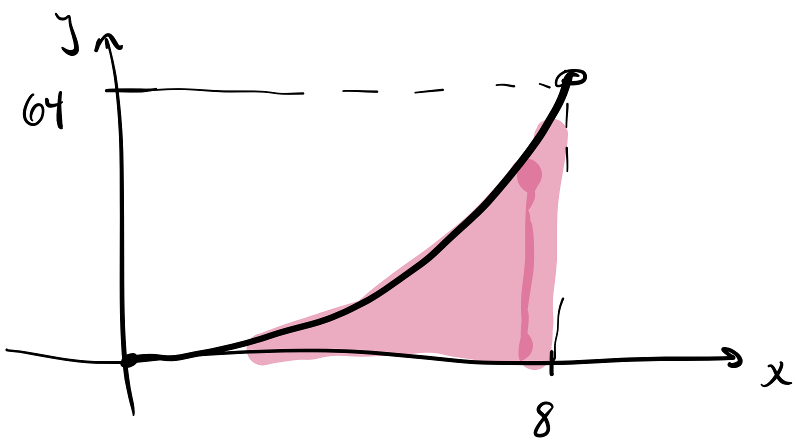
Ex: $\int_0^5 x\,dx = \quad = \frac{1}{2}bh = \frac{1}{2}(5)(5) = \boxed{12.5}$



Ex: $\displaystyle\int_0^{10} 4\,dx = \boxed{40}$



Ex: $\displaystyle\int_0^{8} x^2\,dx = 170.\overline{6}$



Properties of the Definite Integral:

1.) $\displaystyle\int_a^a f(x)\,dx = 0$ (no width)

2.) $\displaystyle\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

3.) Sum/Diff of two Areas:

$$\int_a^b (f(x) \pm g(x))dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$$

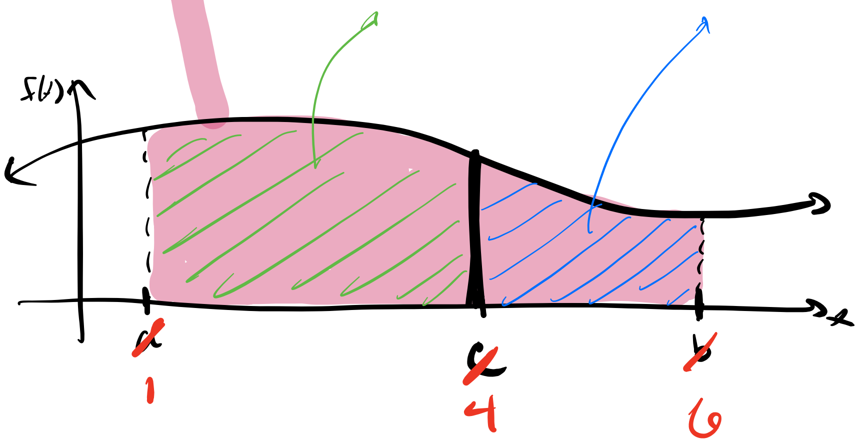
4.) Const. Mult. Rule:

$$\int_a^b c f(x)dx = c \int_a^b f(x)dx$$

$$\int_a^b \underbrace{f(x) + \cdots + f(x)}_{c \text{ times}} dx = \underbrace{\int_a^b f(x)dx + \cdots + \int_a^b f(x)dx}_{c \text{ times}}$$

5.) Any integral can be "split" up into atleast two other areas: Let c be a # between a and b:

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

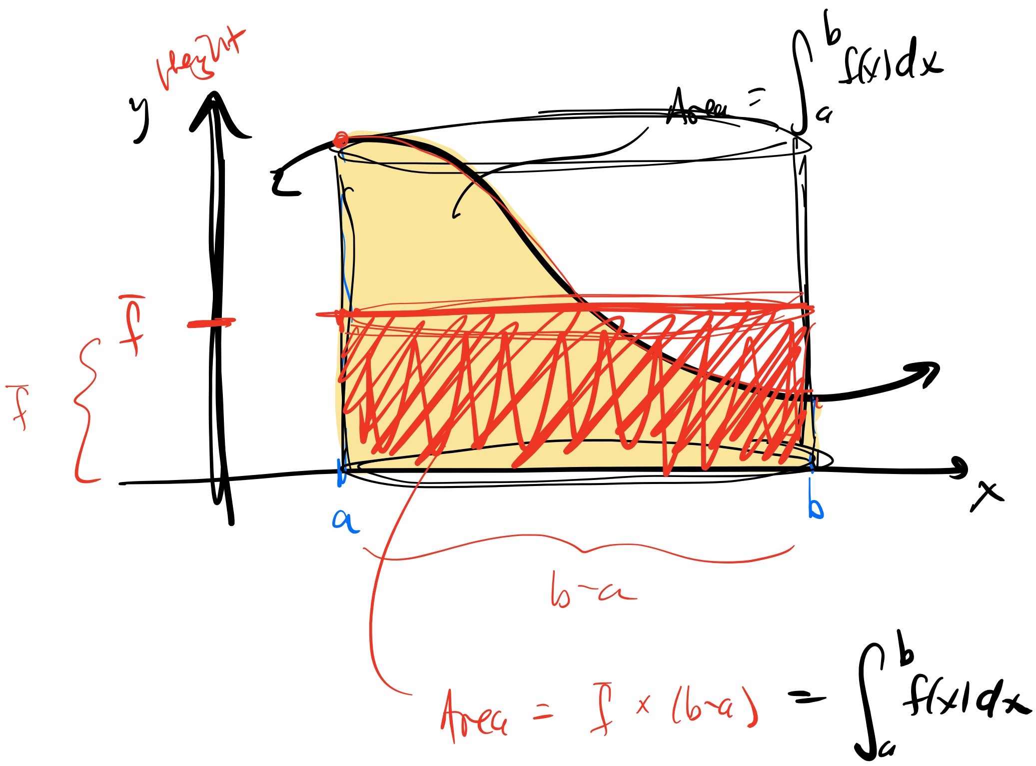


$$\int_1^6 f(x)dx - \int_1^4 f(x)dx = \int_4^6 f(x)dx$$

Def: The average value of a function over the interval $[a,b]$, denoted by $\bar{f}$, is given by

$$\bar{f} = \frac{1}{b-a} \cdot \int_a^b f(x)\,dx$$

Proof:



Recap: $\int_a^b f(x)\,dx$ = "the signed area bounded between the curve $y=f(x)$ and the x-axis on the interval $[a,b]$."

Section 5.3: The Fundamental Theorem of Calculus.

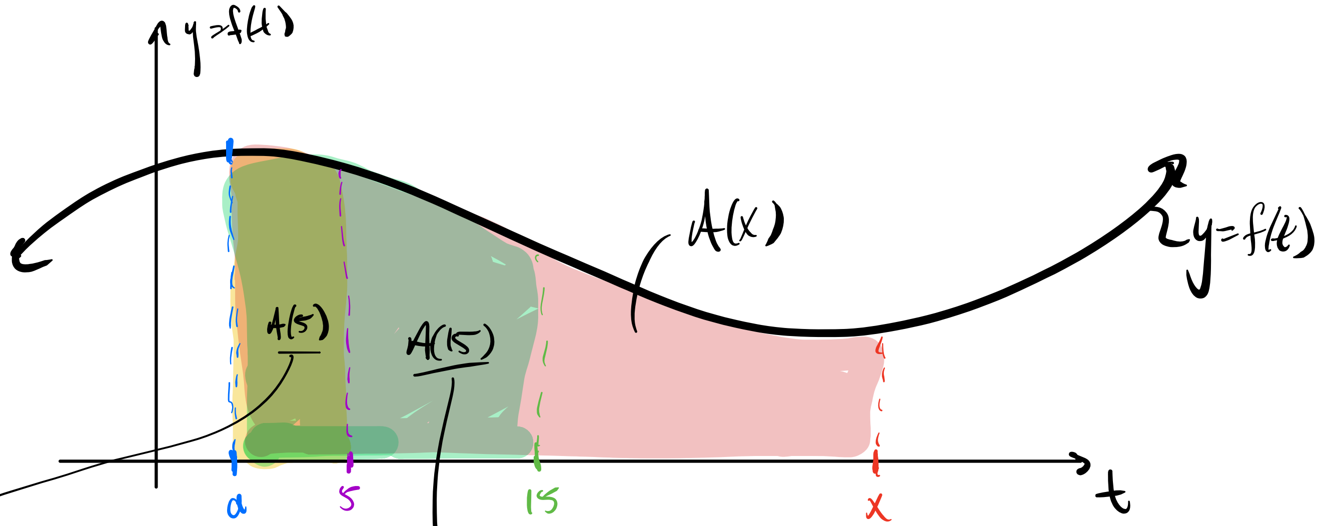
- Before we start, we define an area function:

$$A(x) = \int_a^x f(t)\,dt$$

x = ind. var. of A

t = dummy var. of integration; ind. var. of f

- The function A outputs the net area "under" $f(t)$ over $[a, x]$:



$$A(a) = \int_a^a f(t)\,dt = 0$$

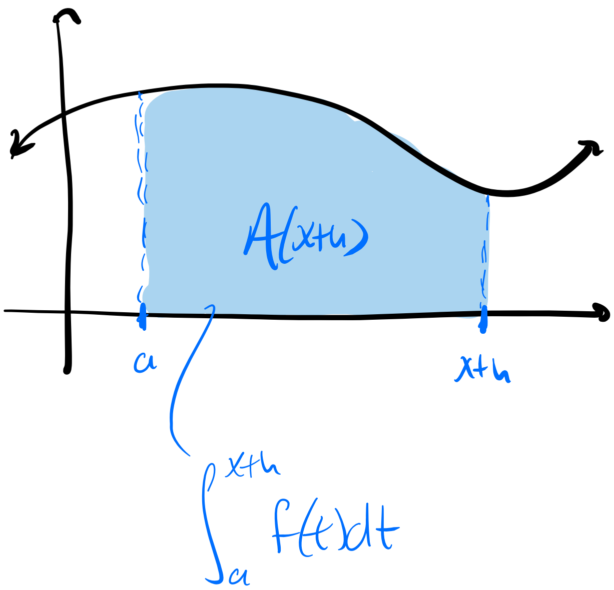
$$A(15) = \int_a^{15} f(t)\,dt = ?? > 0$$

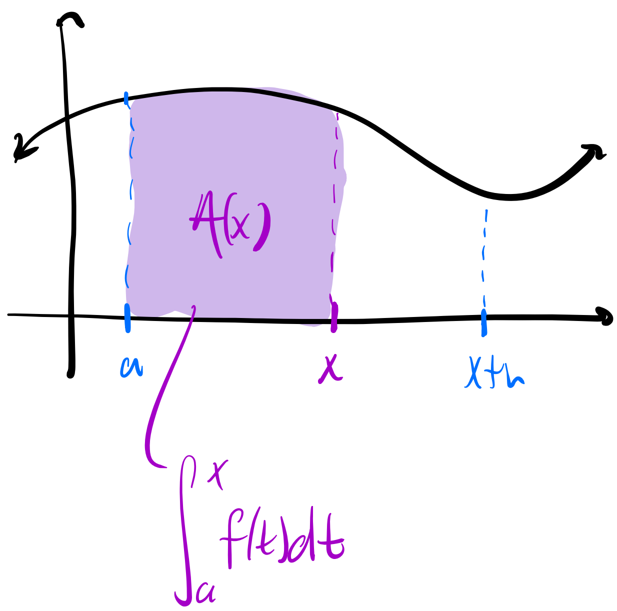
$$A(5) = \int_a^5 f(t)\,dt = ?? > 0$$

FTC, Part 1: The area function $A(x) = \int_a^x f(t)\,dt$ is an anti-derivative of $f(x)$ i.e.

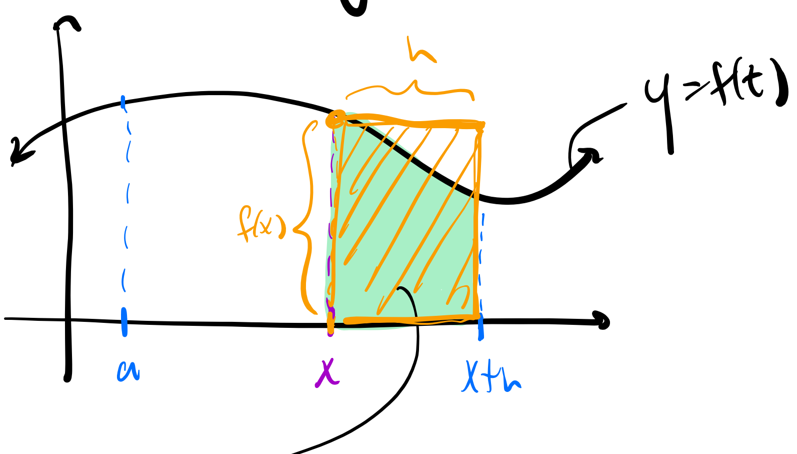
$$A'(x) = f(x).$$

Proof: Consider $A(x)$ and $A(x+h)$, where $h > 0$:





So $A(x+h) - A(x)$ is given by the following:



$$A(x+h) - A(x) \approx f(x) \cdot h$$

This approximation becomes *exact* as $h \to 0$. We have

$$A(x+h) - A(x) \approx f(x) \cdot h$$

$$\Rightarrow \quad \frac{A(x+h) - A(x)}{h} \approx f(x)$$

$$\Rightarrow \quad \lim_{h \to 0} \frac{A(x+h) - A(x)}{h} = \lim_{h \to 0} f(x)$$

$$A'(x) = f(x)$$

and so $A(x)$ is an antiderivative of $f(x)$. ∎

This theorem says that the derivative "undoes" the integral i.e. they are inverse operations.

$$\frac{d}{dx}\left(\underbrace{\int_a^x f(t)\,dt}_{A(x)}\right) = f(x)$$

Ex: Let $A(x) = \int_a^x (t^2 + 3t + 1)\,dt$. Then $A'(x) = x^2 + 3x + 1$.

FTC, Part 2: If $F(x)$ is any antiderivative of $f(x)$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Proof: Let $A(x) = \int_a^x f(t)\,dt$ be the area function. $A(x)$ is an antiderivative of $f(x)$. All antiderivatives differ by a constant. Any antiderivative, $F(x)$, has the form

$$F(x) = \int_a^x f(t)\,dt + C$$

Then

$$F(b) - F(a) = \left[\int_a^b f(t)\,dt + C\right] - \left[\int_a^a f(t)\,dt + C\right]$$

$$= \int_a^b f(t)\,dt + \cancel{C} - \cancel{\int_a^a f(t)\,dt}^{\,0} - \cancel{C}$$

$$= \int_a^b f(t)\,dt. \quad \blacksquare$$

Note: $$\underbrace{F(b) - F(a)}_{} = F(x)\Big|_a^b$$

"$F(x)$ evaluated at b minus $F(x)$ evaluated at a."

Does this make sense?

$$\underbrace{\int_a^b v(t)\,dt}_{\substack{\text{Total dist traveled}\\ \text{over the time interval}\\ t=a \text{ to } t=b.}} = \underbrace{s(b)}_{\substack{\text{Distance}\\ \text{at } t=b}} - \underbrace{s(a)}_{\substack{\text{Dist}\\ \text{at } t=a}}$$

velocity

Position/Dist. function

Section 5.4: Indefinite Integral & u-substitution

Recall: The notation for the derivative:

Newton $\cdot\ f'(2) =$

Leibniz $\cdot\ \left.\dfrac{dy}{dx}\right|_{x=2} =$

"the derivative (slope) at $x=2$"

Newton $\cdot\ f'(x) =$

Leibniz $\cdot\ \dfrac{dy}{dx} =$

"the derivative function"

Mix $\cdot\ \dfrac{d}{dx}(f(x)) =$ "do the derivative of $f(x)$"

- Notation for integration:

Definite Integral

- $$\int_a^b f(x)\,dx = \text{“net area under } f(x) \text{ on } [a,b]\text{”}$$
$$= F(b) - F(a)$$

Indefinite Integral

- $$\int f(x)\,dx = \text{“the anti-derivative function of } f(x)\text{”}$$
$$= F(x) + C$$

- Change of Variables: method of u-substitution is essentially the "reverse chain rule."

Recall: Basic Integration is essentially noticing that the integrand is a derivative i.e.

$$\int f'(x)\,dx = f(x) + C$$

U-Sub: used on integrands of the following form:

$$\int f'(\underbrace{g(x)}_{u})\,\underbrace{g'(x)\,dx}_{du} = f(g(x)) + C.$$

$$u = g(x) \Rightarrow \frac{du}{dx} = g'(x) \Rightarrow du = g'(x)\,dx$$

$$\int f'(u)\,du = f(u) + C$$