MAT 181 - Calculus I
Brooks Emerick

CHAPTER 4 - Applications of Differentiation

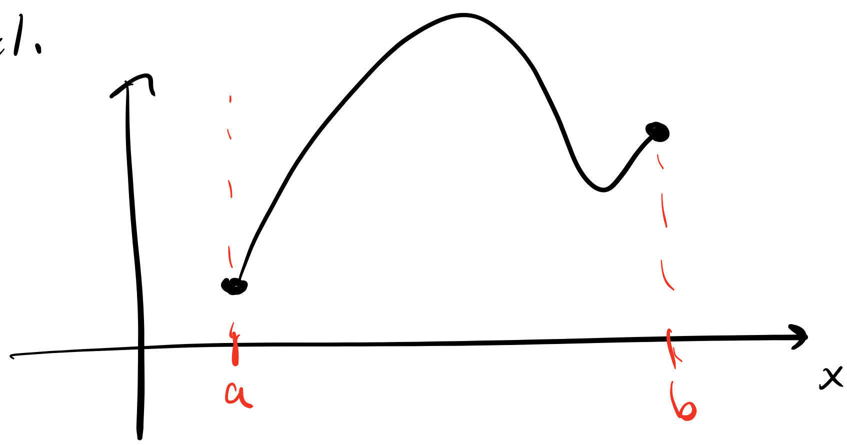
Section 4.1: Extreme values of Functions

(global)

Def: A function of $f(x)$ has an absolute maximum (absolute minimum) value of $f(c)$ at $x=c$ if for every x in an $\boxed{\text{interval}}$ containing c, $f(c) \geq f(x)$ ($f(c) \leq f(x)$).

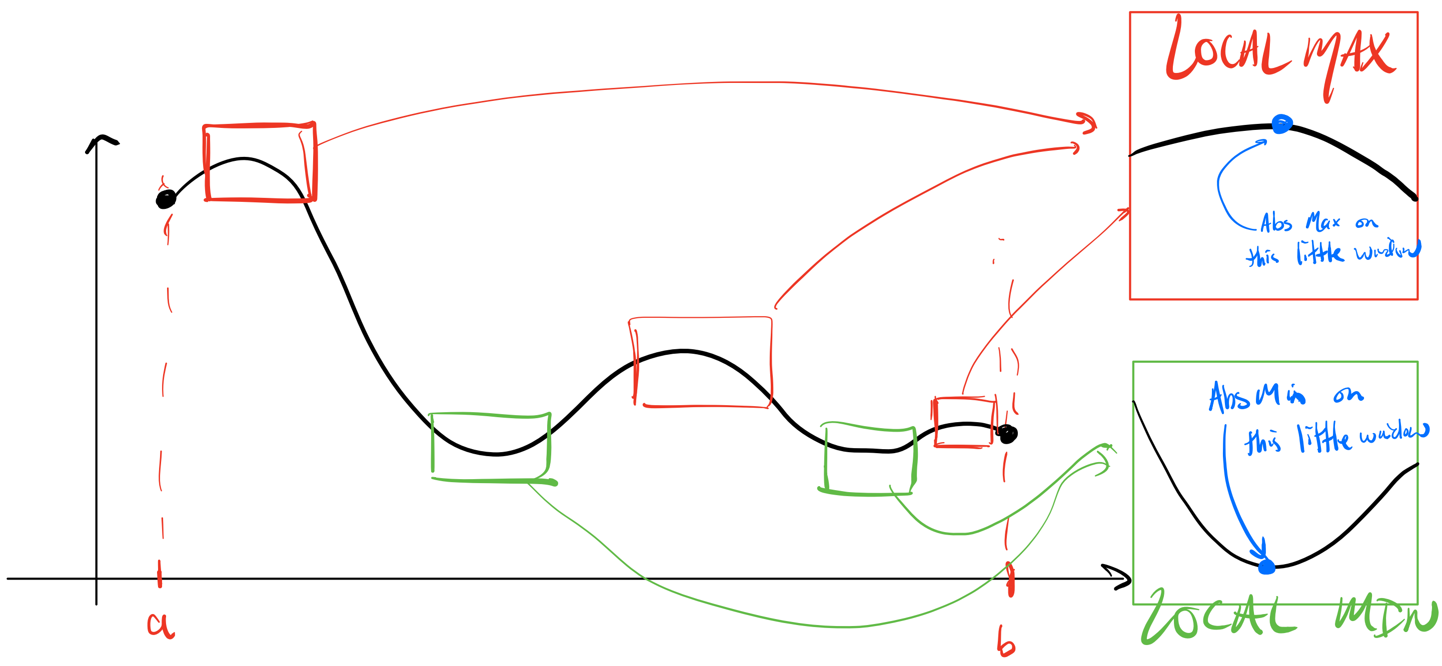
max/min values depend on $f(x)$ as well as the interval of interest

Thm: (Extreme Value Theorem) A function that is cont. on a closed interval $[a,b]$ has an abs. max and an abs. min on that interval.



- **Def**: Let c be in the interior of some interval. If $f(c) \geq f(x)$ ($f(c) \leq f(x)$) for all x in an open interval containing c, then $f(c)$ is a local maximum (local minimum) value.

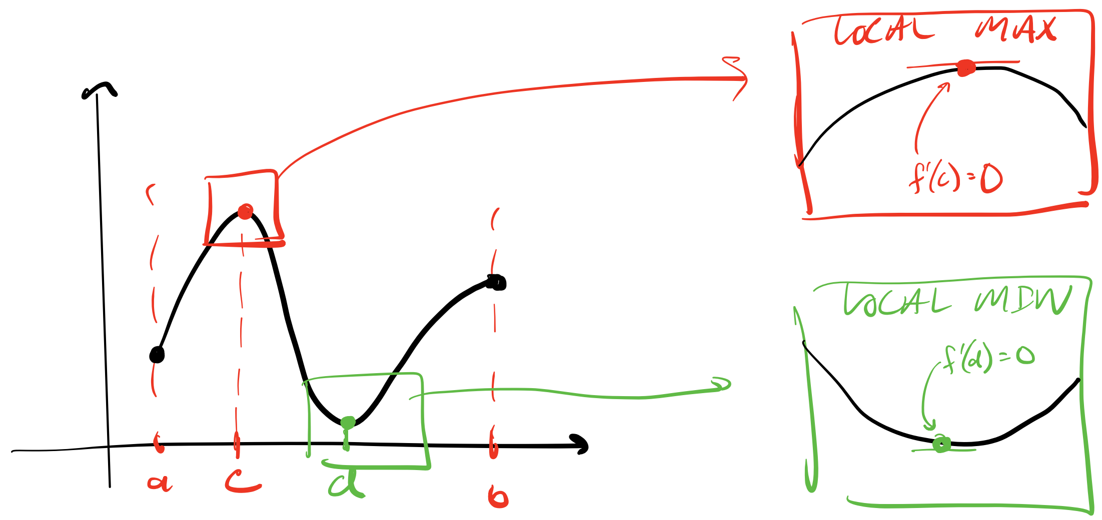
(relative) (relative)



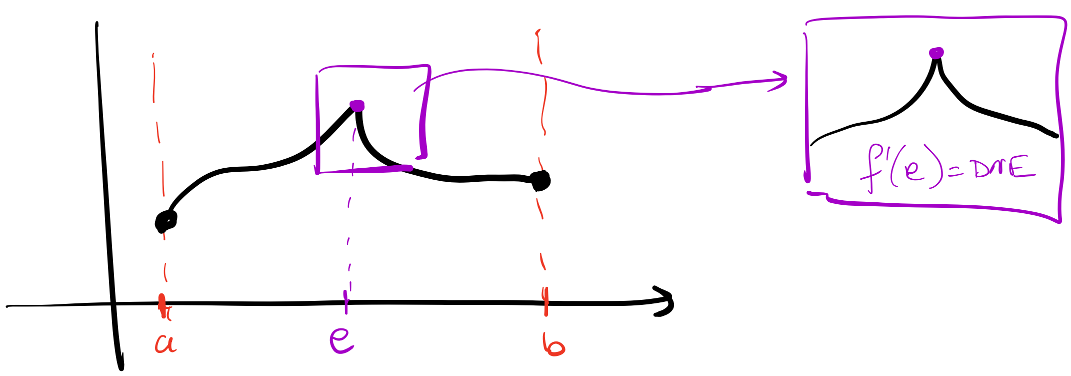
Note: local Extrema cannot occur at the endpoints!

(max/min's

- How do we find local extrema?!



wherever $f'(x)=0$, there could be a local extrema.



wherever the derivative is undefined, there may be a local max/min.

Def: An interior point c in the domain of $f(x)$ at which $f'(c)=0$ or $f'(c)=$ DNE is called a critical point

usually just the x-coordinate.

Thm: (Fermat's Thm) If f has a local extreme point at c inside the interval (a,b) and $f'(c)$ exists, then $f'(c)=0$.

- This theorem together with the extreme value thm, leads to the closed interval method of finding absolute extrema on a closed interval.

Let f be cont. on $[a,b]$. Then to determine the abs max and abs min, follow these steps:

1.) Find all critical points in (a,b).

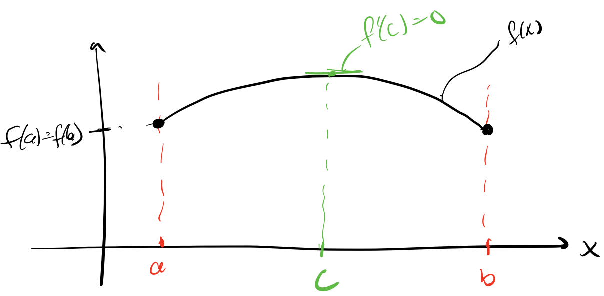
2.) Evaluate the function at the critical pts and at the endpoints (i.e. a and b).

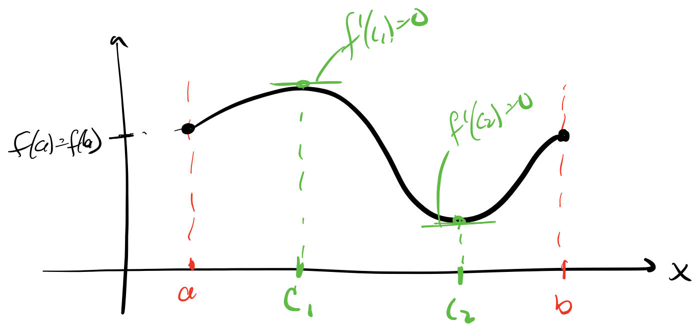
3.) The largest value $\longrightarrow$ abs max.
The smallest value $\longrightarrow$ abs min.

Section 4.2: Mean Value Theorem

- Thm applies to any cont. function on a closed interval $[a,b]$ that is also diff. on the interior (a,b).

- **Thm:** (Rolles' Thm) let $f(x)$ be cont. on $[a,b]$ and diff. on (a,b). If $f(a)=f(b)$, then there is at least one point c inside (a,b) such that $\underline{f'(c)=0}$.





Thm: (Mean Value Theorem) let $f(x)$ be cont. $[a,b]$ and diff. on (a,b). Then there exists at least one value c inside (a,b) such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

Instantaneous Rate of Change

Average Rate of Change

Section 4.3: First and Second Derivative Tests

- The 1st & 2nd Derivative Tests are used to locate max/min (Extreme) values of a function, intervals of inc/dec, and intervals of concave up/down.

$[a,b]$ (a,b)

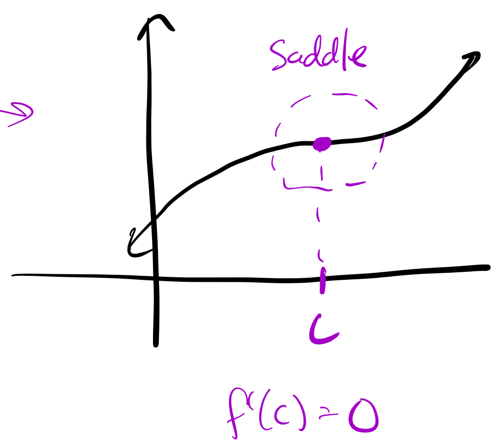
Thm: Suppose f is cont. on an interval $[a,b]$ and diff. on the interior. If $f'(x)>0$ for all x inside the interior, then $f(x)$ is increasing on the interior. Likewise, if $f'(x)<0$, then $f(x)$ is decreasing.

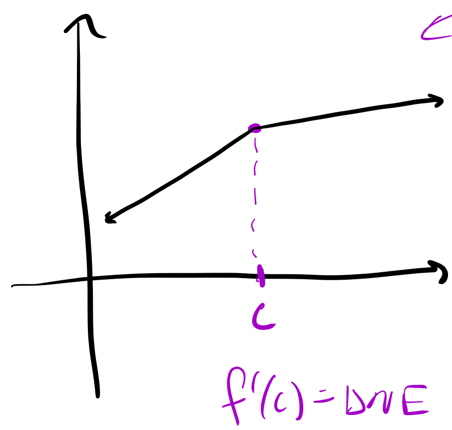
Thm: (First Derivative Test) Suppose f is cont. on an interval containing c and c is a critical point.

1.) If $f'(x)$ changes from <u>pos.</u> to <u>neg.</u> as x increases through c, then $f(x)$ has a <u>local max</u> at c

2.) If $f'(x)$ changes from <u>neg.</u> to <u>pos.</u> as x increases through c, then $f(x)$ has a <u>local min</u> at c.

3.) If $f'(x)$ doesn't change signs, then $f(x)$ has no local extrema at c.



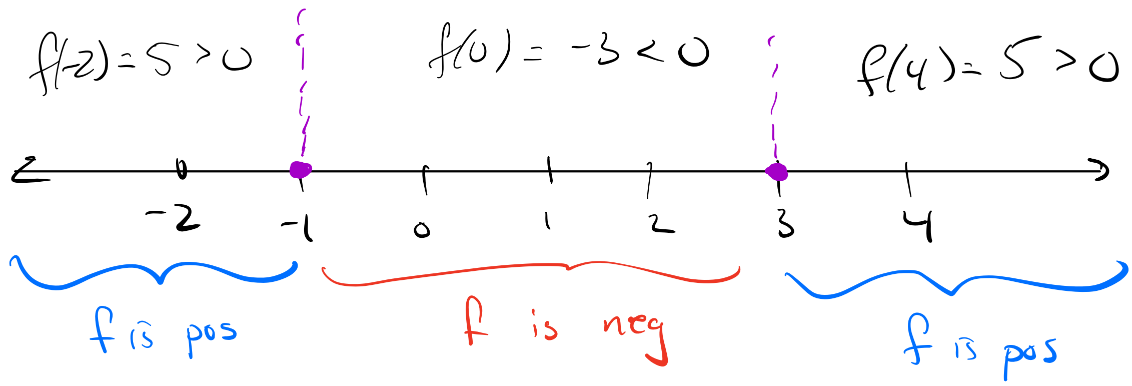


• To use this theorem, it's best to do a <u>sign analysis</u> on the first derivative $f'(x)$.

<u>Ex</u>: Where is $f(x)=x^2-2x-3$ pos. and neg.?

$f(x) = (x-3)(x+1) = 0 \longleftarrow$ find the x-ints

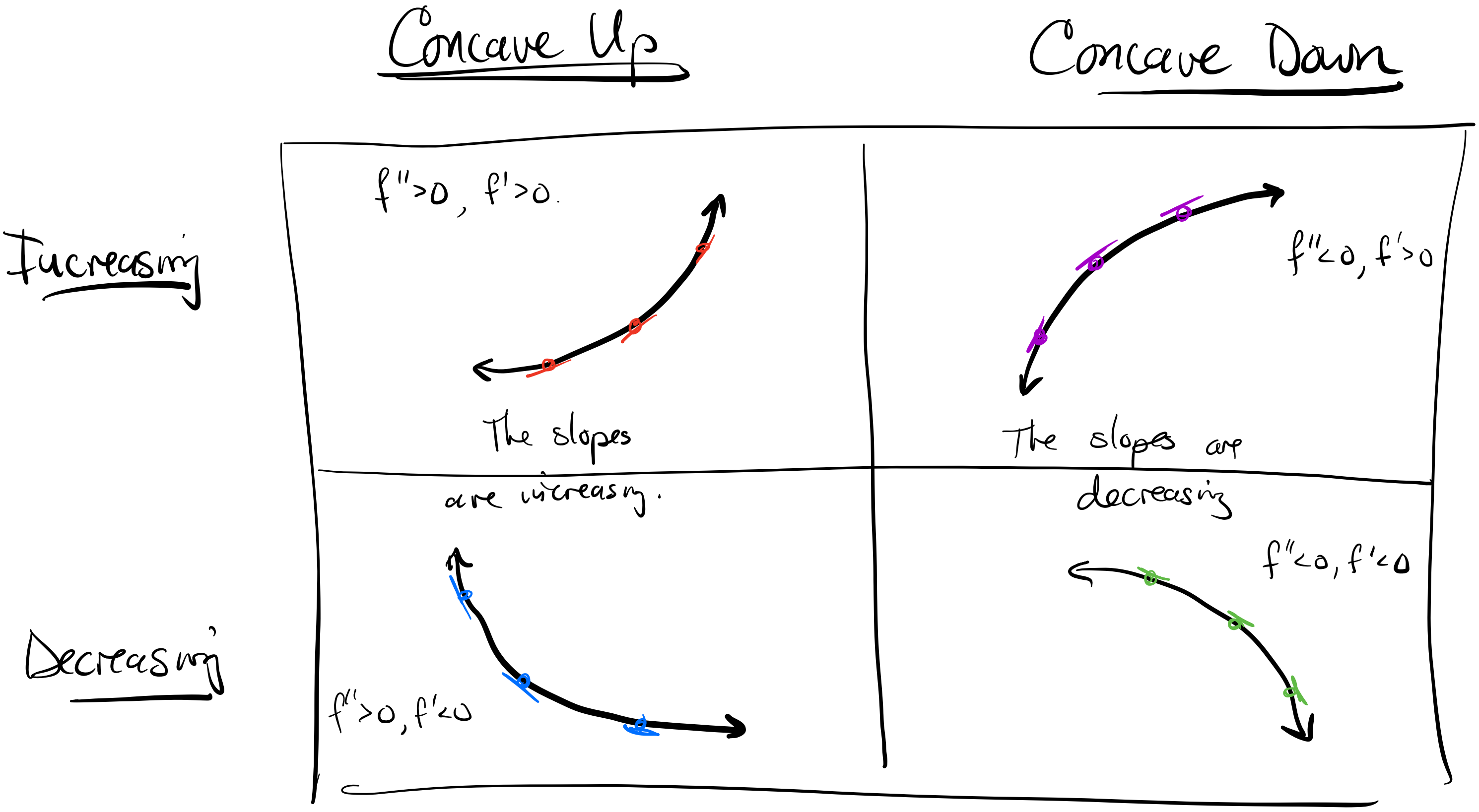
$\Rightarrow \quad x = -1, 3$



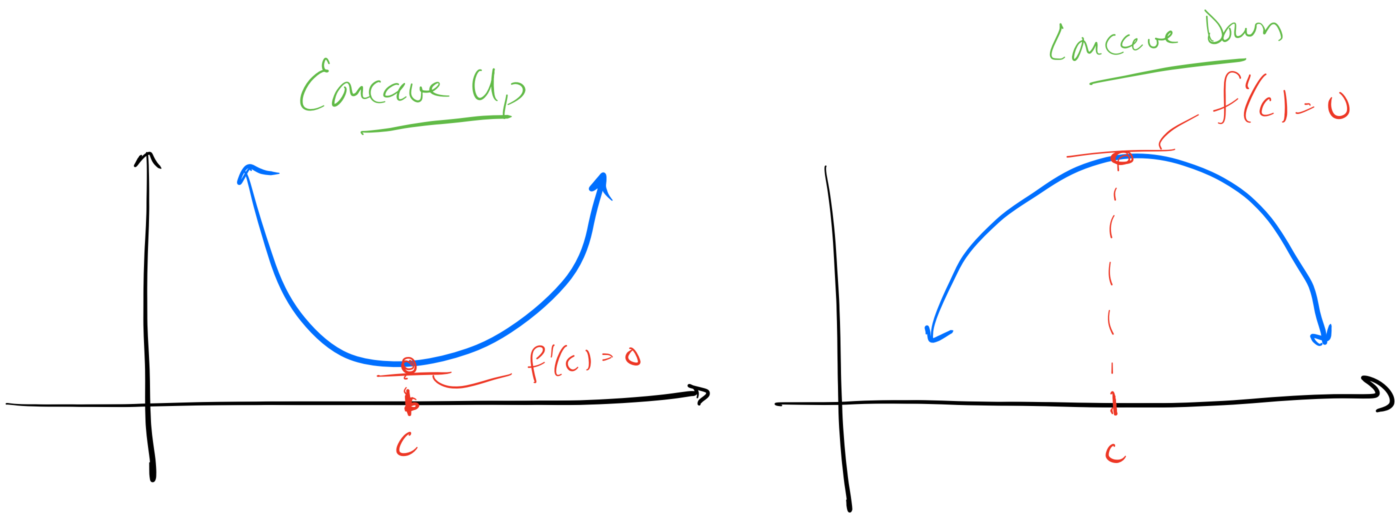
- negative on $(-1, 3)$
- positive on $(-\infty, -1) \cup (3, \infty)$

Recap: If $f'(x) > 0$ (<0) on (a,b), then $f(x)$ is increasing (decreasing) on (a,b).

Def: If $f''(x) > 0$, then $f(x)$ is concave up. If $f''(x) < 0$, then $f(x)$ is concave down.



• Just like the first derivative can be used to determine the location of local extrema, the second derivative can be used determine if critical points satisfying $f'(c)=0$ are local max/min's



$f''(c) > 0$

Local Min

$f''(c) < 0$

Local Max.

Thm: (Second Derivative Test) Suppose $f''(x)$ is cont. on an open interval containing c such that $f'(c) = 0$.

1.) If $f''(c) > 0$, then f has a local min at c.

2.) If $f''(c) < 0$, then f has a local max at c.

3.) If $f''(c) = 0$, then the test is inconclusive (c could be a local max, min, or saddle)

Section 4.4: L'Hôpital's Rule

• Recall from Chapter 2 that our most powerful tool for evaluating limits was continuity:

Ex: $\lim\limits_{x \to 1} \frac{(x+1)^2}{x+2} = \frac{(1+1)^2}{1+2} = \frac{4}{3}$

If the function is defined $\longrightarrow$ plug in the value.

• But if the function isn't defined at the value, we use algebra or the Brute Force Method:

Algebra Ex: $\lim\limits_{x \to 1} \frac{x^2 - 2x + 1}{x - 1} = \lim\limits_{x \to 1} \frac{(x-1)\cancel{(x-1)}}{\cancel{x-1}} = \lim\limits_{x \to 1} x - 1 = 0$

Plug in the value $= \boxed{\frac{0}{0}}$

Indeterminate Form

BFM Ex: $\lim\limits_{x \to 0} \frac{\sin(x)}{x} =$ "Brute Force method" $= 1$

• Limits of the form "$\frac{\pm\infty}{\pm\infty}$" or "$\frac{0}{0}$" are called indeterminate.

Thm: (L'Hôpital's Rule) Suppose f & g are differentiable at c and suppose $\lim_{x\to c} f(x) = 0$ $(\pm\infty)$ and $\lim_{x\to c} g(x) = 0$ $(\pm\infty)$. Then

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}$$

provided that the limit on the right exists (or is $\pm\infty$).

This rule also applies to $x\to c^+$, $x\to c^-$, or $x\to\pm\infty$.

∘ A few strange examples:

Ex: $\lim_{x\to\infty} \frac{3x^4 - x}{6x^4+12} = \infty \cdots$ L'H$\cdots = \frac{1}{2}$

Another way: $\lim_{x\to\infty} \frac{\frac{3x^4}{x^4} - \frac{x}{x^4}}{\frac{6x^4}{x^4} + \frac{12}{x^4}} = \lim_{x\to\infty} \frac{3 - 1/x^3}{6 + 12/x^4} = \frac{3-0}{6+0} = \frac{3}{6} = \boxed{\frac{1}{2}}$

Ex: $\lim_{x\to\infty} \frac{3x^3 - x}{6x^4+12} = \lim_{x\to\infty} \frac{3/x - 1/x^3}{6 + 12/x^4} = \frac{0-0}{6+0} = \frac{0}{6} = 0$

Divided every term by x^4

Ex: $\lim_{x\to\infty} \frac{3x^4 - x}{6x^3+12} = \lim_{x\to\infty} \frac{3x - 1/x^2}{6 + 12/x^3} = \frac{\infty - 0}{6+0} = \boxed{\infty}$

Divided every term by x^3

- A limit may take the form of an <u>indeterminate</u> <u>product</u> or an <u>indeterminate</u> <u>difference</u>:

"$(\pm\infty)\cdot(0)$" — product

Rewrite the product as a quotient

Then use L'Hopital's

"$\infty - \infty$" — Difference

Rewrite as quotient by getting a common denominator

Use L'Hopital's Rule

- Consider a function of the form $f(x)^{g(x)}$. For example:

$$x^x, \quad \left(1+\frac{1}{x}\right)^x, \quad x^{\frac{1}{x}}$$

These types of functions may lead to indeterminate powers:

$$\text{"}1^\infty\text{"}, \quad \text{"}0^0\text{"}, \quad \text{"}\infty^0\text{"}$$

- The trick is to put it into quotient form:

$$f(x)^{g(x)} \longrightarrow e^{\ln\left(f(x)^{g(x)}\right)} \longrightarrow e^{g(x)\ln(f(x))}$$

$$\lim_{x\to c} f(x)^{g(x)} \longrightarrow \lim_{x\to c} e^{\ln(f(x)^{g(x)})} \longrightarrow e^{\lim_{x\to c} g(x)\ln(f(x))}$$

Section 4.6: Optimization

$\underbrace{\text{Optimization}}$ → Maximize, Minimize

Operations Research 361/362

- Optimization is a wide reaching application that is used in many disciplines. History reaches back to World War II.

- Characteristic of all optimization problems is the idea of an objective function with some constraints:
  - Obj. fun.: the function we seek to maximize/minimize (e.g. max profit, max revenue, min costs, ...)
  - Constraints: guidelines or limits to the problem that must be satisfied.

Ex: Find two non-neg numbers x & y $\underbrace{\text{that sum up to 20}}_{\text{Constraint}}$ and have $\underbrace{\text{the largest product}}_{\text{Obj. fun}}$.

Brute Force:

x	y	xy
✓ 10	10	100
× 9	11	99
× 12	8	96
× π	$20-\pi$	$(20-\pi)\pi$
× 1	19	19

Set up Problem

Obj Fun: Maximize product; $P = xy \xrightarrow{\text{update}} P(x) = x(20-x)$

Constraints: $x \geq 0, y \geq 0$, $x+y=20$

$\boxed{y = 20-x}$

Apply Calcul

Eliminate y, find $P(x)$ and $P'(x)$

$$P(x) = x(20-x)$$
$$\Rightarrow P(x) = 20x - x^2$$
$$\Rightarrow P'(x) = 20 - 2x$$

→ Find Critical Pts →

Critical Pts

$$20 - 2x = 0$$
$$\Rightarrow 2x = 20$$
$$\Rightarrow \boxed{x = 10}$$

$y = 20 - 10$

$\boxed{y = 10}$ (From Constraint)

check 2nd Derivative Test

Confirming it's a max/min

$$P''(x) = -2$$
$$P''(10) = -2 < 0 \rightarrow \boxed{x=10 \text{ is a max}}$$

Section 4.7: Antiderivatives.

- **Recall**: we've been given a function $f(x)$ and we've been asked to determine $f'(x)$. The reverse of this process is called "antidifferentiation".

- **Def**: A function $F(x)$ is an antiderivative of $f(x)$ if

$$F'(x) = f(x)$$

on some interval.

Ex: let $\boxed{f(x) = 1}$, then $\boxed{F(x) = x}$ is an antiderivative.

Ex: let $\boxed{f(x) = 2x}$, then $\boxed{F(x) = x^2 + 10}$ is an antiderivative.

Ex: let $\boxed{f(x) = \frac{1}{1+x^2}}$, then $\boxed{F(x) = \arctan(x)}$ is an antiderivative.

- The antiderivative is not unique. In fact for any function $f(x)$, there is a family of antiderivatives:

$$\vdots$$

$$F(x) = x - 100$$

$$F(x) = x - 50 \xrightarrow{F'} f(x) = 1$$

$$F(x) = x \xrightarrow{F'} f(x) = 1 \longrightarrow \text{Thus, we write:}$$

$$F(x) = x + 2 \xrightarrow{F'} f(x) = 1$$

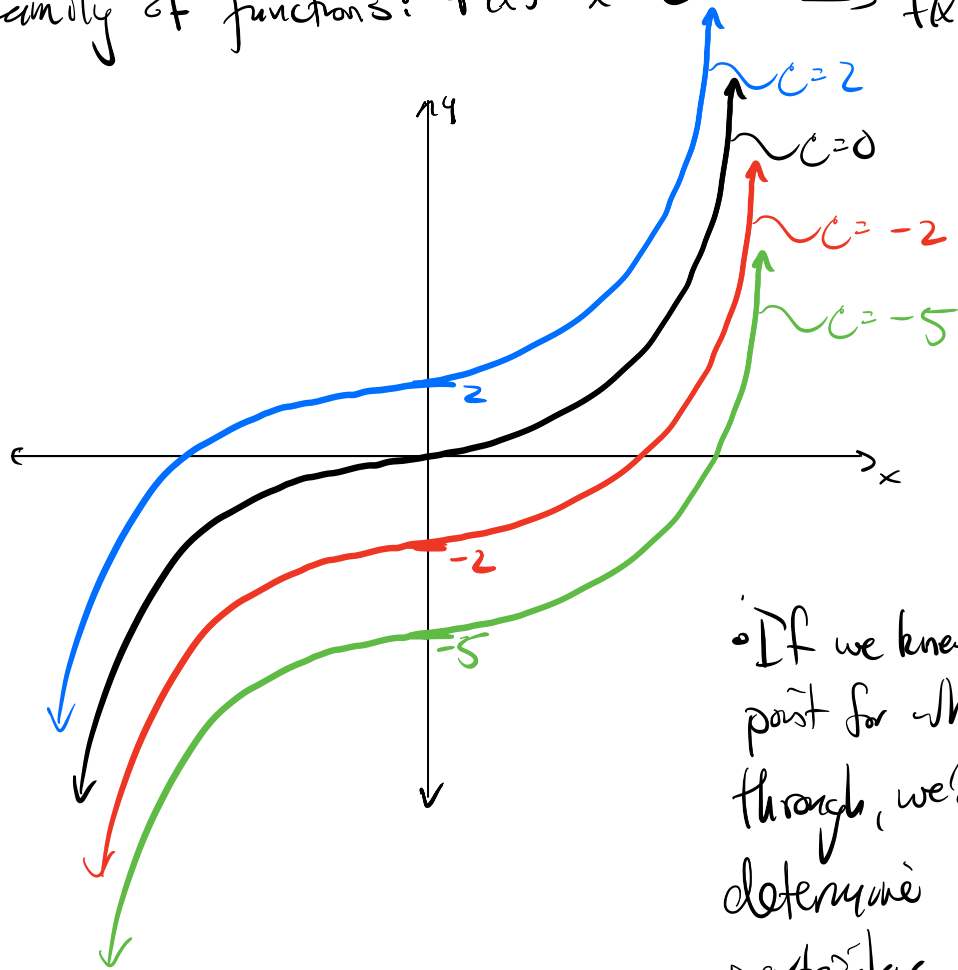
$$F(x) = x + \pi \xrightarrow{F'} f(x) = 1$$

$$\vdots$$

$$\boxed{F(x) = x + C \text{ is the general antiderivative}}$$

where C is any real #

• Consider the family of functions: $F(x) = x^3 + C$ $\longrightarrow$ $f(x) = 3x^2$



• If we knew one additional point for which $F(x)$ passes through, we'd be able to determine C — the <u>particular solution</u>.

Ex: Let $f(x) = \cos(x)$, then $F(x) = \sin(x) + C$.

Let $f(x) = e^x$, then $F(x) = e^x + C$.

Let $f(x) = \ln(3) \cdot 3^x$, then $F(x) = 3^x + C$.

<u>Thm</u>: (Reverse Power Rule) Let $f(x) = x^p$ for $p \neq -1$, then the antiderivative is

$$F(x) = \frac{1}{p+1} x^{p+1} + C.$$

Ex: Let $f(x) = x^2$, then $F(x) = \frac{1}{3}x^3 + C$

Check: $F'(x) = \frac{1}{3}(3x^2) = x^2$

Ex: Let $f(x) = x^4$, then $F(x) = \frac{1}{5}x^5 + C$.

Ex: Consider the following list

$$F(x) = e^x + C$$

$$f'(x) = e^x + c$$

$$F(x) = e^x + C$$

$$f(x) = e^x$$

$$f'(x) = e^x$$

$$f''(x) = e^x$$

$$f'''(x) = e^x$$

$$f^{(4)}(x) = e^x$$

$$\downarrow$$

Differentiation

Antidifferentiation

Ex: Let $f(x) = x^{-1} = \frac{1}{x}$, then $F(x) = \ln(x) + C$.

Eg: Let $f(x) = x^{\pi}$, then $F(x) = \frac{1}{\pi+1} x^{\pi+1} + C$.

Thm: (Exp. Functions) Let $f(x) = b^x$, then the antiderivative is

$$F(x) = \frac{1}{\ln(b)} \cdot b^x + C$$

- A differential equation is an equation involving a function and its derivatives.

general solutions

(Easy)

Ex: $f'(x) = 3x^2 - 2x + 5$

Anti derivative

$$f(x) = x^3 - x^2 + 5x + C$$

~~(Hard)~~

~~**Ex:** $f'(x) = f(x)$~~

~~$f(x) = Ae^x$~~

- An Initial Value Problem (IVP) is a diff. equ. coupled with initial data.

particular sols.

Ex: $f'(x) = 3x^2 - 2x + 5$, $f(1) = 5$

Anti Diff

$$f(x) = x^3 - x^2 + 5x + C$$

$$f(1) = 1^3 - 1^2 + 5(1) + C$$

$\Rightarrow \quad 5 = 1 - 1 + 5 + c$

$\Rightarrow \quad 0 = c$

$\Rightarrow \quad \boxed{f(x) = x^3 - x^2 + 5x}$ Particular Sol.