• To derive the derivatives of the other trig functions, we can use identities and the quotient rule.

<u>Thm</u>: (Derivatives of other trig functions)

$$\frac{d}{dx}\left(\tan(x)\right) = \sec^2(x) \qquad \frac{d}{dx}\left(\cot(x)\right) = -\csc^2(x)$$

$$\frac{d}{dx}\left(\sec(x)\right) = \sec(x)\tan(x) \qquad \frac{d}{dx}\left(\csc(x)\right) = -\csc(x)\cot(x)$$

<u>Proof</u>: We'll prove the derivative of $\cot(x)$. Consider

$$\begin{aligned}
\frac{d}{dx}\left(\cot(x)\right) &= \frac{d}{dx}\left(\frac{\cos(x)}{\sin(x)}\right) \\
&= \frac{\sin(x)\left(-\sin(x)\right) - \cos(x)\left(\cos(x)\right)}{\sin^2(x)} \\
&= \frac{-\sin^2(x) - \cos^2(x)}{\sin^2(x)} \\
&= \frac{-\left(\sin^2(x) + \cos^2(x)\right)}{\sin^2(x)} \\
&= \frac{-1}{\sin^2(x)} \\
&= -\cot^2(x). \quad \blacksquare
\end{aligned}$$

$\boxed{\text{WS}}$ #2 – #3 working with derivatives.

• Don't forget that on the way to deriving the derivatives of sine and cosine, we established the following limits:

$$\lim_{x \to 0} \frac{\cos(x)-1}{x} = 0 \quad \text{and} \quad \lim_{x \to 0} \frac{\sin(x)}{x} = 1$$

$\boxed{\text{WS}}$ #4 Evaluating special limits.

Section 3.4: Chain Rule

- We now formulate a derivative technique that can be used for composite functions of the form $f(g(x))$.
- Consider the function $y = f(g(x))$. The "inside" function is $g(x)$ and the "outside" function is $f(x)$. Let $u = g(x)$, then $y = f(u)$. We see that f is a function of u and u is a function of x. This means

$$y = f(u) = f(g(x)).$$

Here, y is a function of x but only through composition via u. Hence, we differentiate y with respect to x in the following manner using Leibniz notation:

$$\frac{dy}{dx} = \underbrace{\frac{dy}{du}}_{f'(u)} \cdot \underbrace{\frac{du}{dx}}_{u'(x)}$$

Substituting for u, we have the following differentiation rule in Newton's notation:

$$\left(f(g(x))\right)' = f'(g(x)) \cdot g'(x)$$

Essentially, we differentiate the outside function first, leaving the inside portion alone, then multiply by the derivative of the inside function.

Thm: (Chain Rule) If $g(x)$ is differentiable function, and $f(x)$ is differentiable at $g(x)$, then $(f \circ g)(x) = f(g(x))$ is also differentiable with

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$$

or in Leibniz notation, letting $y = f(u)$ and $u = g(x)$,

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

WS #1 - #5 work with using the chain rule.

Section 3.5: Implicit Differentiation.

Typically when we define a function in terms of the variable x, we plot that function on the y-axis versus the x-axis by writing $y = f(x)$ to represent the graph. This means that y is defined explicitly in terms of x. Further we know $y = f(x)$ passes the vertical line test.

- Not all equations that relate two variables are defined explicitly. In fact, we know the following relationship maps a circle of radius one in the x-y plane:

$$x^2+y^2=1$$

This relationship is defined implicitly in the variables x and y. We cannot actually represent the entire circle as a single function, but we can try:

$$x^2+y^2=1 \Rightarrow y^2=1-x^2 \begin{cases} \rightarrow y=\sqrt{1-x^2}=f_T(x) & \text{Top of circle} \\ \rightarrow y=-\sqrt{1-x^2}=f_B(x) & \text{Bottom of circle.} \end{cases}$$

- Here, we can use either the "top" function or the "bottom" function to find the slope of the curve:

$$f_T'(x)=\frac{1}{2\sqrt{1-x^2}}\cdot -2x = \frac{-x}{\sqrt{1-x^2}}$$ ← slope at points on top of circle

$$f_B'(x)=\frac{-1}{2\sqrt{1-x^2}}\cdot(-2x) = \frac{x}{\sqrt{1-x^2}}$$ ← slope at points on bottom of circle.

- It may not always be the case that we're able to extract certain functions that represent a portion of the curve.
- For example consider

$$x+y^3-x^2y=1$$

This is very difficult to solve for y explicitly in terms of x. Hence, to find the slope, we can use <u>implicit differentiation</u>:

- Here is an example: Consider again the unit circle:

$$x^2+y^2=1$$

Differentiate the entire equation

$$\Rightarrow \frac{d}{dx}\left(x^2+y^2\right)=\frac{d}{dx}(1)$$

$$\Rightarrow \frac{d}{dx}(x^2)+\frac{d}{dx}(y^2)=\frac{d}{dx}(1)$$

$$\Rightarrow 2x+2y\cdot\frac{dy}{dx}=0$$

For the term involving y^2, we must use the chain rule when differentiating y with respect to x.

$$\Rightarrow 2y\cdot\frac{dy}{dx}=-2x$$

$$\Rightarrow \frac{dy}{dx}=\frac{-2x}{2y}$$

$$\Rightarrow \boxed{\frac{dy}{dx}=-\frac{x}{y}}$$

note that $\frac{dy}{dx}$ depends on y as well.

- The key to this technique is to always remember that when differentiating any terms involving y, we must multiply by $\frac{dy}{dx}$ because of the chain rule. This is because y is always (or assumed to be) a function of x.

- Here are a few examples:

$$\frac{d}{dx}(x)=1 \qquad \frac{d}{dx}(y)=\frac{dy}{dx}$$

$$\frac{d}{dx}(x^2) = 2x \qquad \frac{d}{dx}(y^2) = 2y \cdot \frac{dy}{dx}$$

$$\frac{d}{dx}(x^3) = 3x^2 \qquad \frac{d}{dx}(y^3) = 3y^2 \cdot \frac{dy}{dx}$$

$$\frac{d}{dx}(e^x) = e^x \qquad \frac{d}{dx}(e^y) = e^y \cdot \frac{dy}{dx}$$

- Here are a few other examples:

Product Rule.

$$\frac{d}{dx}(xy) = \frac{d}{dx}(x) \cdot y + x \cdot \frac{dy}{dx} = y + x\frac{dy}{dx}$$

$$\frac{d}{dx}(xy^2) = \frac{d}{dx}(x) \cdot y^2 + x \cdot \frac{d}{dx}(y^2) = y^2 + x \cdot 2y\frac{dy}{dx}$$

$\boxed{\text{WS}}$ #1 – #3 working with implicit differentiation

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Section 3.6: Derivatives of Inverse Functions

- In this section, we derive formulas for the derivatives of logarithm functions and the inverse trig functions using implicit differentiation.

- **Thm**: (Derivative of logarithms) Let $f(x) = \log_b(x)$ where $b > 1$, then $f'(x) = \frac{1}{\ln(b) \cdot x}$ or

$$\frac{d}{dx}(\log_b(x)) = \frac{1}{\ln(b) \cdot x}$$

Proof: We can prove this using implicit differentiation. Let $y = \log_b(x)$, then we seek $\frac{dy}{dx}$. First, by the definition of the logarithm we can write

$$y = \log_b(x) \Rightarrow b^y = b^{\log_b(x)} \Rightarrow b^y = x.$$

Now, we can differentiate

$$\frac{d}{dx}(b^y) = \frac{d}{dx}(x)$$

$$\Rightarrow \ln(b) \cdot b^y \cdot \frac{dy}{dx} = 1$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\ln(b) \cdot b^y}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\ln(b) \cdot x},$$

since $b^y = x$

as desired.

Corollary: (Derivative of Natural log) Let $f(x) = \ln(x)$, then $f'(x) = \frac{1}{x}$

or

$$\frac{d}{dx}(\ln(x)) = \frac{1}{x}$$

Proof: Let $b = e$ in the formula above, then $f(x) = \log_e(x) = \ln(x)$ and so

$$\frac{d}{dx}(\ln(x)) = \frac{1}{\ln(e) \cdot x} = \frac{1}{x}. \blacksquare$$

$\boxed{\text{WS}}$ #1 working with the derivatives of logs.

- Using an identical method, we can derive a formula for the inverse trig functions.
- Recall that for inverse trig functions:

$$y = \sin(x) \iff x = \arcsin(y)$$
$$y = \cos(x) \iff x = \arccos(y)$$
$$y = \tan(x) \iff x = \arctan(y).$$

The notation $\arcsin(x)$ means "inverse sine," which is a function that "undoes" the action of sine. Likewise for the others.

<u>Thm</u>: (Derivative of Inverse Trig Functions) The following derivatives hold:

$$\frac{d}{dx}(\arcsin(x)) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\arccos(x)) = -\frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\arctan(x)) = \frac{1}{1+x^2}$$

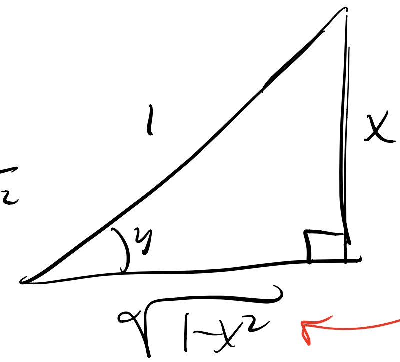
<u>Proof</u>: We will prove the derivative of $\arcsin(x)$. The others are proved in a similar way. Let $y = \arcsin(x)$, then we seek $\frac{dy}{dx}$. If $y = \arcsin(x)$, then we know $\sin(y) = x$. This gives the following via implicit differentiation:

$$\sin(y) = x$$
$$\Rightarrow \frac{d}{dx}(\sin(y)) = \frac{d}{dx}(x)$$
$$\Rightarrow \cos(y)\frac{dy}{dx} = 1$$
$$\Rightarrow \frac{dy}{dx} = \frac{1}{\cos(y)}.$$

We need to convert $\cos(y)$ back into terms of x. Consider the following triangle given by the relation $\sin(y) = x$:

$$\sin(y) = x = \frac{x}{1} \left(\frac{\text{opp}}{\text{Hyp}}\right)$$

$$\cos(y) = \frac{\sqrt{1-x^2}}{1} = \sqrt{1-x^2}$$



Found using Pythagorean's Thm

Therefore, we can substitute $\sqrt{1-x^2}$ in for $\cos(y)$ yielding

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

as desired. ∎

$\boxed{\text{WS}}$ #2 - #4 Working with inverse trig functions.

Section 3.7: Rates of Change in Use.

- One of the most basic applications of the derivative is population growth.
- To model the growth or decay of a concentration/population we assume that its rate of change (ie. derivative) is proportional to the current population size. First define our variables:
    - t = time (days, minutes, months, ...)
    - $P(t)$ = population density/concentration at time t.

- Our model takes the form

$$\underbrace{P'(t)}_{\text{Rate of change}} = \underbrace{k}_{\text{Proportionality Constant}} \underbrace{P(t)}_{\text{Current Population or Concentration.}}$$

We can "solve" this differential equation for $P(t)$. What function has a derivative that is a constant multiple times itself?

Ex: $k=1$: $P'(t) = P(t) \rightarrow P(t) = e^{t}$

$k=2$: $P'(t) = 2P(t) \rightarrow P(t) = e^{2t}$

$k=k$: $P'(t) = kP(t) \rightarrow P(t) = e^{kt}$!

- In fact, the most general function is

$$\boxed{P(t) = P_0 e^{kt}}$$

where P_0 is the initial population. We can show that this function does indeed satisfy the equation:

$$P'(t) = (P_0 e^{kt})' = P_0 (e^{kt})' = P_0 (k e^{kt}) = k \underbrace{P_0 e^{kt}}_{P(t)} = kP(t).$$

- For any two pieces of data, we can formulate the function of interest. (two points on the graph of $P(t)$.)

ws #1 walk with a population growth function.

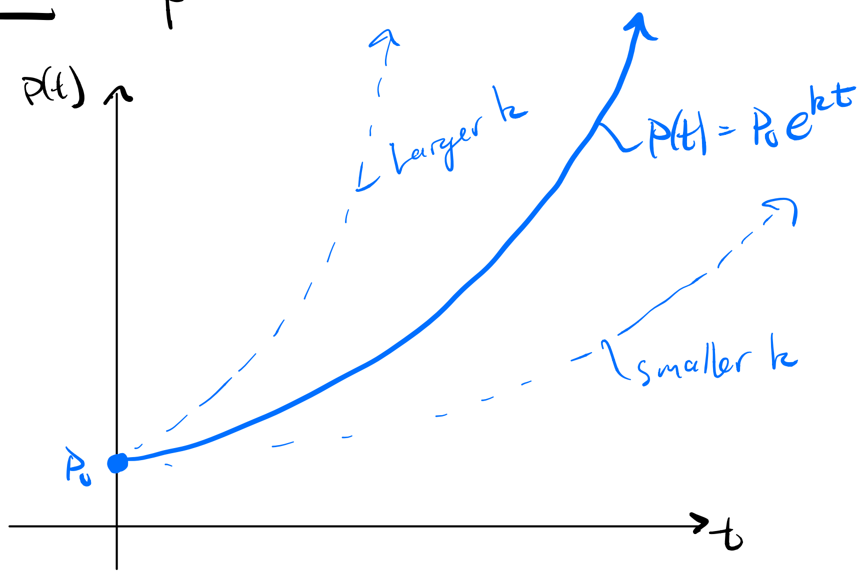
- The function $P(t) = P_0 e^{kt}$ has the special property that $P'(t) = kP(t)$. The sign of k denotes whether the population is growing or decaying:

- **$\underline{k>0}$: Population Growth**

Examples: cells, ants, rabbits, savings account.

$$\lim_{t\to\infty} P_0 e^{kt} = \infty$$

(grows without bound)



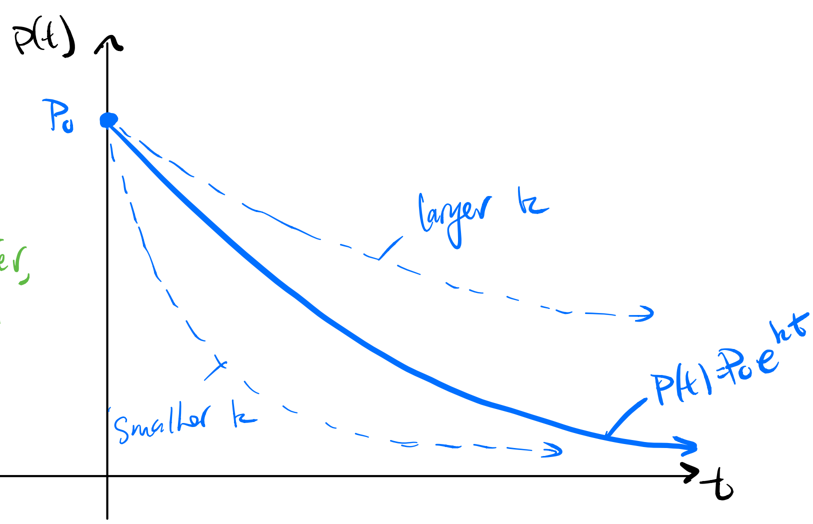
- k is the per capita growth rate.

- **$\underline{k<0}$: Population Decay**

Examples: concentration of a radioactive material, fluid exiting a container, decaying chemical concentration.

$$\lim_{t\to\infty} P_0 e^{kt} = 0$$

(decays to 0)



- k is the per capita decay rate.

$\boxed{\text{WS}}$ #2 work with a decaying population model.

- One slight variation of the population decay model is called Newton's Law of Cooling:

- let t be time (e.g. hours, minutes, seconds...)
- let $T(t)$ be the temperature of a substance
- let T_s be the temperature (constant) of the surroundings.

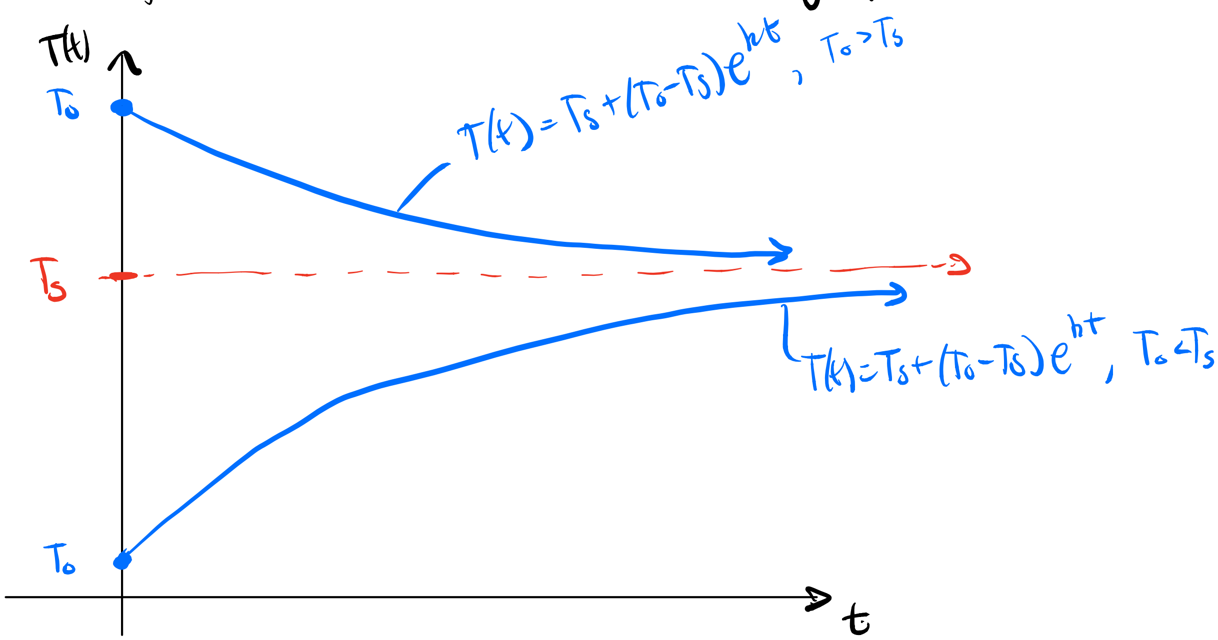
- This model assumes that the rate of change of the temperature, $T'(t)$, is proportional to the difference between the current temperature and the temperature of the surroundings. i.e.

$$T'(t) = k(T(t) - T_s)$$

Solution: $T(t) = T_s + (T_0 - T_s)e^{kt}$, where $k<0$.

- Graphically, this function has a horizontal asymptote at T_s:

If $T_0 > T_s$, the temp will lower to room temp



If $T_0 < T_s$, the temperature will rise to room temp.

In both cases, $\lim_{t \to \infty} T(t) = T_s$ i.e. the temp eventually reaches that of the surroundings.

$\boxed{\text{WS}}$ #3 work with a Newton's Law of Cooling Example