MAT 181 – Calculus
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CHAPTER 3 - Differentiation

Section 3.1: Differential Notation

- **Def:** A function is differentiable at c if the derivative $f'(c)$ exists i.e. if the limit

$$f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

exists.

- To differentiate a function is to derive its derivative function. There are many ways to denote the derivative function.

1.) $\underbrace{f'(x)} = \lim_{h \to 0} \frac{f(x+h) - f(\ )}{h}$

↳ Prime Notation; created by Isaac Newton

2.) $\underbrace{\frac{df}{dx}} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

First to discover

↳ Differential Notation; created by Gottfried Leibniz

- Leibniz' notation is more informative in the sense that it reminds us that the derivative is a difference quotient:

$$\left[\text{Avg Rate of Change}\right] = \frac{f(x+h) - f(x)}{h} \longleftrightarrow \frac{\text{change in } f}{\text{change in } x} = \frac{\Delta f}{\Delta x}$$

(no limit involved)

$$\left[\text{Inst Rate of Change}\right] = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} = \frac{df}{dx}$$

- When we perform the limit, the "Δ" becomes a "d".

- There are variations in both notations; since $y=f(x)$ in most applications, we have

$$f'(x) = y' = y'(x) = \frac{df}{dx} = \frac{dy}{dx} = \underbrace{\frac{d}{dx}(y) = \frac{d}{dx}(f(x))}_{\text{means "do the derivative"}} = \underbrace{D(f(x)) = D_x(f(x))}_{\text{rare}}$$

All of these are different representations of the derivative function.

- The notation for the derivative at a point $x=c$:

1.) $f'(c) = \lim_{h\to 0} \frac{f(c+h)-f(c)}{h}$ ← Newton's

2.) $\left.\frac{df}{dx}\right|_{x=c} = \lim_{h\to 0} \frac{f(c+h)-f(c)}{h}$ ← Leibniz. ← more clunky

$\boxed{\text{WS}}$ #1 Find the derivative and use both notations

- In some applications, taking the derivative of a function more than once is necessary. (e.g. The derivative of position is velocity. The derivative of velocity is acceleration. Thus, the second derivative of position is acceleration.)
- Notation for the second derivative:

1.) $(f'(x))' = \boxed{f''(x)} = \lim_{h\to 0} \frac{f'(x+h)-f'(x)}{h}$

2.) $\frac{d}{dx}\left(\frac{dy}{dx}\right) = \boxed{\frac{d^2y}{dx^2}} = \lim_{h\to 0} \frac{f'(x+h)-f'(x)}{h}$.

- Notation for the nth derivative:

1.) $f^{\overbrace{''''''}^{n \text{ ticks}}}(x) = \boxed{f^{(n)}(x)}$

2.) $\underbrace{\frac{d}{dx}\left(\frac{d}{dx}\left(\cdots\left(\frac{d}{dx}\right.\right.\right.}_{n \text{ times}}\left.\left.\left.\frac{dy}{dx}\right)\right)\right) = \boxed{\frac{d^ny}{dx^n}}$

WS #2 working with multiple derivatives

- We can talk a little more thoroughly about what it means for a function to be differentiable at a point, say c. In fact, when is a function not differentiable?

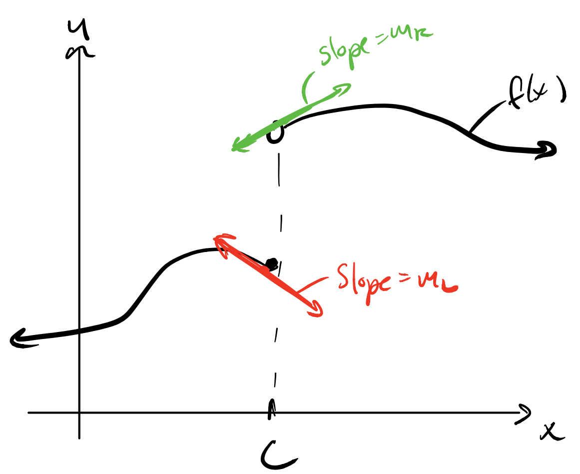
$$\text{Differentiable at } c \iff \text{there exists a unique tangent line to } f \text{ at } c.$$

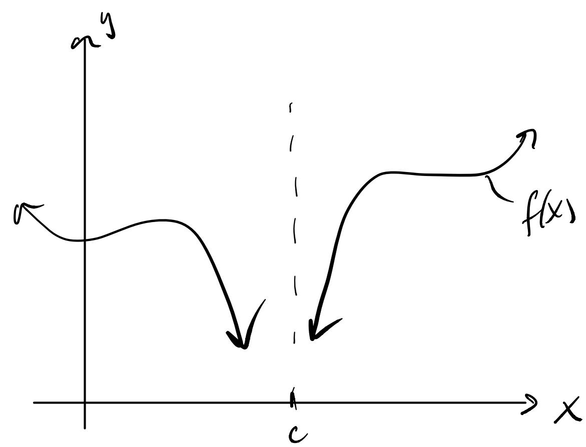
- First off, a function being differentiable at c automatically means that it is continuous at c. Being differentiable is harder to obtain than be continuous.

- This doesn't mean that all continuous functions are differentiable!!

- However, if a function is <u>not continuous</u> at c, then it is <u>not differentiable</u> at c. Let's look at some examples:

Not Continuous $\xrightarrow{\text{automatically means}}$ **Not Differentiable**





Slope on left = m_L
Slope on right = m_R } Clearly not the same

Tangent line doesn't exist at c!
$\Rightarrow$ Not differentiable

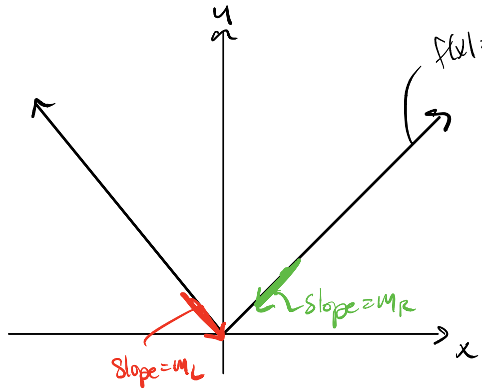
- $f(c)$ doesn't exist.
- How can we compute $\lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$

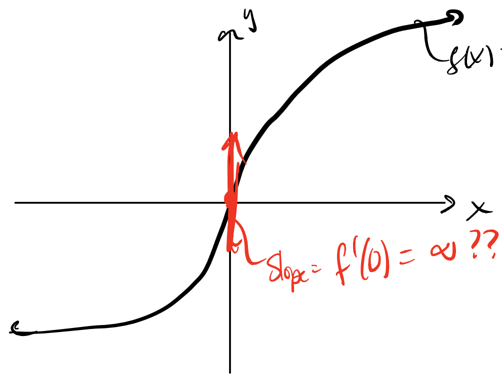
DNE!

Not differentiable.

BE AWARE: A function can be continuous, but not differentiable!

• Consider the two functions functions $f(x) = |x|$ and $f(x) = x^{1/3}$.





• f is cont everywhere.
• At $x=0$, there is a corner
• At $x=0$, the slopes are different!!

$m_L \neq m_R$

• f is cont everywhere
• At $x=0$, the function has an infinite slope!

Not Differentiable!

• In short, if the function comes to a corner/cusp at c or if the slope is infinite, then the function is not differentiable at c.

• Conclusion: A function f is not differentiable at $x=c$ if at least one of the following conditions hold.

1.) f is not continuous at $x=c$
2.) f has a vertical tangent at $x=c$.
3.) f has a corner or cusp at $x=c$.

• In the case when f has a cusp at $x=c$, we can examine the slope on either side of c by defining the left and right derivatives

· **Def**: The <u>right-hand derivative</u> at c is given by

$$f_+'(c) = \lim_{h\to 0^+} \frac{f(c+h)-f(c)}{h} \quad \text{or} \quad f_+'(c) = \lim_{x\to c^+} \frac{f(x)-f(c)}{x-c}$$

Likewise, the <u>left-hand derivative</u> at c is given by

$$f_-'(c) = \lim_{h\to 0^-} \frac{f(c+h)-f(c)}{h} \quad \text{or} \quad f_-'(c) = \lim_{x\to c^-} \frac{f(x)-f(c)}{x-c}$$

$\boxed{\text{WS}}$ #3 work with one-sided derivatives to show f is not differentiable at c.

<u>Section 3.2</u>: Derivatives of Polynomials, Exponentials, Products, and Quotients.

• In this section, we derive some basic differentiation rules that will essentially make our lives much easier.

<u>Thm</u>: (Constant Rule) Let f be a constant function, $f(x)=k$, then $f'(x)=0$ or in Leibniz notation

$$\frac{df}{dx} = \frac{d}{dx}(k) = 0.$$

<u>Proof</u>: Let $f(x)=k$, then $f(x+h)=k$, so that

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} \frac{k-k}{h} \\ &= \lim_{h\to 0} 0 \\ &= 0. \end{aligned}$$

Hence, $f'(x)=0$. ∎

<u>Thm</u>: (Positive Integer Power Rule) Suppose $f(x) = x^n$, where n is any positive integer, then $f'(x) = nx^{n-1}$ or

$$\frac{df}{dx} = \frac{d}{dx}(x^n) = nx^{n-1}.$$

<u>Proof</u>: Let $f(x) = x^n$, then

$$f(x+h) = (x+h)^n = x^n + \overbrace{nx^{n-1}h + \cdots + nxh^{n-1} + h^n}^{\text{Every term has at least one } h.}$$

we have

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{\cancel{x^n} + nx^{n-1}h + \cdots + nxh^{n-1} + h^n - \cancel{x^n}}{h} \\ &= \lim_{h\to 0} \frac{nx^{n-1}h + \cdots + nxh^{n-1} + h^n}{h} \\ &= \lim_{h\to 0} \frac{\cancel{h}(nx^{n-1} + \cdots nxh^{n-2} + h^{n-1})}{\cancel{h}} \\ &= \lim_{h\to 0} nx^{n-1} + \underbrace{\cdots + nxh^{n-2} + h^{n-1}}_{\text{All have an } h!} \\ &= nx^{n-1} \end{aligned}$$

Thus, $f'(x) = nx^{n-1}$. ∎

$\boxed{\text{WS}}$ # 1 work with basic derivatives.

- <u>Thm</u>: (Constant Multiple Rule) Let k be any constant. Then, if f is differentiable, we have that the derivative of $kf(x) = kf'(x)$, or

$$\frac{d}{dx}\left(kf(x)\right) = k\frac{d}{dx}(f(x)) = k\frac{df}{dx}.$$

- <u>Thm</u> (Sum/Diff Rule) If f and g are both differentiable, then the sum/diff $f(x) \pm g(x)$ is also differentiable, $(f \pm g)'(x) = f'(x) \pm g'(x)$ or

$$\frac{d}{dx}(f(x) \pm g(x)) = \frac{df}{dx} \pm \frac{dg}{dx}.$$

• With the above two theorems, we can differentiate polynomials.

$\boxed{WS}$ #2 work with differentiating polynomials

• Another important derivative is the derivative of the exponential function $f(x) = b^x$, where $b > 1$ is the base of the exponential function.

• Note, in the function $f(x) = b^x$, the variable is in the exponent not in the base. Thus, we can not use the power rule:

Power Rule:

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

n = constant exponent

x = variable base

CANNOT USE POWER RULE

$$\frac{d}{dx}(b^x) \neq x b^{x-1}$$

x = variable exponent

b = constant base

• To derive the derivative of the exponential function $f(x) = b^x$, we must use the limit definition of the derivative:

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{b^{x+h} - b^x}{h} \\ &= \lim_{h \to 0} \frac{b^x b^h - b^x}{h} \\ &= \lim_{h \to 0} \frac{(b^h - 1) b^x}{h} \\ &= \underbrace{\left(\lim_{h \to 0} \frac{b^h - 1}{h} \right)}_{\text{constant, what is the value of this limit?}} b^x \end{aligned}$$

$f(x+h) = b^{x+h}$

$f(x) = b^x$

8

• We conclude that the derivative is just some constant times the function b^x:

$$f(x) = b^x \longrightarrow f'(x) = \underbrace{\left(\lim_{h\to 0} \frac{b^h - 1}{h}\right)}_{\text{Let's evaluate this limit for different values of } b.} b^x$$

• Consider the following limit:

$$\lim_{h\to 0} \frac{b^h-1}{h} \longrightarrow b=2: \lim_{h\to 0}\frac{2^h-1}{h} \simeq \frac{2^{.00001}-1}{.00001} \approx 0.6931 = \ln(2)$$

$$b=2.5: \lim_{h\to 0}\frac{(2.5)^h-1}{h} \simeq \frac{(2.5)^{.00001}-1}{.00001} \approx 0.9163 = \ln(2.5)$$

$$b=3: \lim_{h\to 0}\frac{3^h-1}{h} \simeq \frac{3^{.00001}-1}{.00001} \approx 1.0986 = \ln(3)$$

$$b= \ : \lim_{h\to 0}\frac{(3.5)^h-1}{h} \simeq \frac{(3.5)^{.00001}-1}{.00001} \approx 1.2528 = \ln(3.5)$$

Somewhere between 2.5 & 3 the limit is equal to 1! That # is e!

• Take any value b, then it can be shown that

$$\lim_{h\to 0}\frac{b^h-1}{h} = \ln(b).$$

<u>Thm</u>: (Exponential Rule) Let $f(x) = b^x$, then $f'(x) = \ln(b)\cdot b^x$

or

$$\frac{d}{dx}(b^x) = \ln(b)\cdot b^x.$$

Thm: (Natural Exponential Rule) Let $f(x) = e^x$, then $f'(x) = e^x$ or

$$\frac{d}{dx}(e^x) = e^x.$$

Proof: Let $f(x) = e^x$, then using the general exponential rule, we have

$$\begin{aligned} f'(x) &= \ln(e) \cdot e^x \\ &= 1 \cdot e^x \\ &= e^x. \blacksquare \end{aligned}$$

$\boxed{WS}$ #3–#5 Working with derivatives and Tangent lines.

- Another important derivative is the derivative of the product of two functions.
- **Note**: The derivative of a product is <u>not</u> the product of the derivatives, i.e.

$$\frac{d}{dx}\big(f(x) \cdot g(x)\big) \neq f'(x) \cdot g'(x)$$

Thm: (Product Rule) Let $f(x)$ and $g(x)$ be differentiable functions, and suppose the product $(fg)(x)$ is also differentiable, then $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$ i.e.

$$\frac{d}{dx}\big(f(x) \cdot g(x)\big) = \frac{df}{dx} \cdot g(x) + f(x) \cdot \frac{dg}{dx}.$$

Proof: Using the limit definition, we have

$$(fg)'(x) = \lim_{h\to 0} \frac{(fg)(x+h) - (fg)(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h}$$

$$= \lim_{h\to 0} \frac{[f(x+h) - f(x)]g(x+h)}{h} + \lim_{h\to 0} \frac{f(x)[g(x+h) - g(x)]}{h}$$

$$= \left(\lim_{h\to 0} \frac{f(x+h) - f(x)}{h}\right)\left(\lim_{h\to 0} g(x+h)\right) + \left(\lim_{h\to 0} f(x)\right)\left(\lim_{h\to 0} \frac{g(x+h) - g(x)}{h}\right)$$

$$= f'(x)g(x) + f(x)g'(x). \blacksquare$$

$\boxed{\text{WS}}$ #1 working with the product rule.

- Another important rule is the quotient rule i.e. the derivative of a quotient of two functions.

Note: As with products, the derivative of a quotient is <u>not</u> the quotient of the derivatives. i.e.

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) \neq \frac{f'(x)}{g'(x)}.$$

Thm: (Quotient Rule) Let $f(x)$ and $g(x)$ be differentiable functions with $g(x) \neq 0$. Then $(f/g)(x)$ is also differentiable with

$$(f/g)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$$ or

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{\frac{df}{dx}\cdot g(x) - f(x)\cdot\frac{dg}{dx}}{g(x)^2}.$$

Proof: We can use the product rule to prove the quotient rule.

Let $h(x) = \frac{f(x)}{g(x)}$, then we have

$$h = \frac{f}{g} \Rightarrow f = gh$$

$$\Rightarrow f' = (gh)'$$

$$\Rightarrow f' = g'h + gh'$$

$$\Rightarrow f' = g'\left(\frac{f}{g}\right) + gh'$$

$$\Rightarrow gh' = f' - \frac{g'f}{g}$$

$$\Rightarrow h' = \frac{f'}{g} - \frac{g'f}{g^2}$$

$$\Rightarrow h' = \frac{f'g}{g^2} - \frac{g'f}{g^2}$$

$$\Rightarrow h' = \frac{f'g - g'f}{g^2}$$

$$\Rightarrow \left(f/g(x)\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2} \quad \blacksquare$$

$\boxed{\text{WS}}$ #2 - #4 working with quotient rule.

Section 3.3: Derivatives of Trig Functions.

· In this section, we derive the derivatives of the six trigonometric functions.

Thm: (Derivatives of Sine & Cosine) Let $f(x) = \sin(x)$ and $g(x) = \cos(x)$, then $f'(x) = \cos(x)$ and $g'(x) = -\sin(x)$ or

$$\frac{d}{dx}(\sin(x)) = \cos(x) \quad \text{and} \quad \frac{d}{dx}(\cos(x)) = -\sin(x).$$

Proof: we will prove the derivative of $\sin(x)$. First, recall the identity

$$\sin(A+B) = \sin(A)\cos(B) + \cos(A)\sin(B)$$

as it will prove useful. We have

$$\frac{d}{dx}(\sin(x)) = \lim_{h\to 0} \frac{\sin(x+h)-\sin(x)}{h}$$

$$= \lim_{h\to 0} \frac{\sin(x)\cos(h)+\cos(x)\sin(h)-\sin(x)}{h}$$

$$= \lim_{h\to 0} \frac{\sin(x)(\cos(h)-1)+\cos(x)\sin(h)}{h}$$

$$= \sin(x)\underbrace{\left(\lim_{h\to 0}\frac{\cos(h)-1}{h}\right)}_{\text{constant}} + \cos(x)\underbrace{\left(\lim_{h\to 0}\frac{\sin(h)}{h}\right)}_{\text{constant.}}$$

note, if we can evaluate the orange limits above, we're done. Inspecting the graphs of $\frac{\cos(x)-1}{x}$ and $\frac{\sin(x)}{x}$ near $x=0$ yields the following results:

$$\lim_{h\to 0}\frac{\cos(h)-1}{h} = 0 \quad \text{and} \quad \lim_{h\to 0}\frac{\sin(h)}{h} = 1.$$

Thus, we have

$$\frac{d}{dx}(\sin(x)) = \cos(x) \quad \blacksquare$$

The proof for the derivative of $\cos(x)$ is similar.

$\boxed{\text{WS}}$ #1 working with the derivative of sine & cosine.