MAT 106 - Trigonometry
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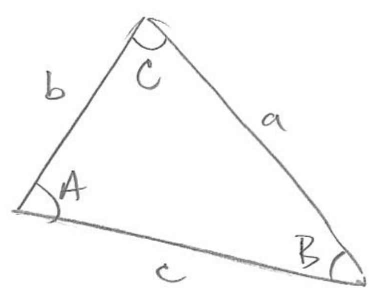
CHAPTER 3 - Applications of Trigonometry

Section 3.1: Law of Sines

- In the previous chapter, we worked exclusively with triangles with a right angle. In this section, we consider the relationships between the angles and sides of any triangle.
- A triangle without any right angles is called an oblique triangle and will be denoted as the following:



Angles: capital letters (A, B, C)
Sides: lowercase letters (a, b, c)

Note: $A + B + C = 180°$

- Side a is located opposite to the angle A.
- Side b is located opposite to the angle B.
- Side c is located opposite to the angle C.

- Solving an oblique triangle means finding the lengths of all three sides and the angular measure of all three angles.
- When dealing with oblique triangles, it is beneficial to use degree measure.
- Recall from basic geometry that to solve an oblique triangle we need three pieces of information, which results in 4 cases:
  1.) Two angles and any side (AAS or ASA)
  2.) Two sides and any angle opposite one of them (SSA)

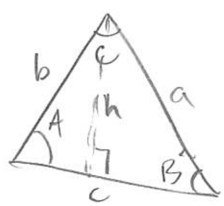
3.) Three Sides (SSS)
4.) Two sides and the angle between them (SAS)

- Parts 1.) and 2.) can be solved using the Law of Sines. Parts 3.) and 4.) can be solved using the Law of Cosines.

- Law of Sines: Let ABC be any triangle with sides a, b, c, then

$$\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)}$$

Proof: Consider the acute oblique triangle with height h:



we know $\sin(A) = \frac{h}{b} \Rightarrow h = b\sin(A)$

Also, $\sin(B) = \frac{h}{a} \Rightarrow h = a\sin(B)$

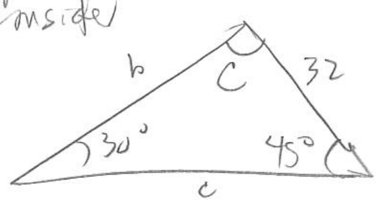
Therefore, $h = h \Rightarrow b\sin(A) = a\sin(B)$

$$\Rightarrow \frac{b}{\sin(B)} = \frac{a}{\sin(A)}$$

A similar argument holds if we draw h from the angle B.

- We first consider the case when we're given AAS:

Ex: Consider



$A = 30°$ $\quad$ $C = ??$
$B = 45°$ $\quad$ $c = ??$
$a = 32$ cm $\quad$ $b = ??$

We can quickly find the remaining angle C.

$$C = 180 - 45 - 30 \Rightarrow \boxed{C = 105°}$$

We can use the Law of Sines to find b and c:

$$\frac{a}{\sin(A)} = \frac{b}{\sin(B)} \qquad \frac{a}{\sin(A)} = \frac{c}{\sin(C)}$$

$$\Rightarrow \frac{32}{\sin(30)} = \frac{b}{\sin(45^\circ)} \qquad \Rightarrow \frac{32}{\sin(30)} = \frac{c}{\sin(105^\circ)}$$

$$\Rightarrow \frac{32}{\frac{1}{2}} = \frac{b}{\frac{1}{\sqrt{2}}} \qquad \Rightarrow \frac{32}{\frac{1}{2}} = \frac{c}{.966}$$

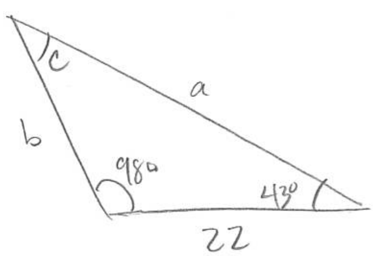
$$\Rightarrow 64\sqrt{2} = b \qquad \Rightarrow 64(.966) = c$$

$$\Rightarrow \boxed{b = 90.510} \qquad \boxed{c = 61.824}$$

$\boxed{\text{WS}}$ #1 Use Law of Sines to solve an AAS problem.

The other type that is easily done is the ASA problem.

Ex:



$A = 98°$ $\quad$ $C = ??$

$B = 43°$ $\quad$ $b = ??$

$c = 22$ ft $\quad$ $a = ??$

Again, we can quickly find the remaining angle:

$$C = 180 - 98 - 43 = 39° \Rightarrow \boxed{C = 39°}$$

We can then use the Law of Sines to find a and b:

$$\frac{c}{\sin(C)} = \frac{b}{\sin(B)} \qquad \frac{c}{\sin(C)} = \frac{a}{\sin(A)}$$

$$\Rightarrow \frac{22}{\sin(39^\circ)} = \frac{b}{\sin(43^\circ)} \qquad \Rightarrow \frac{22}{\sin(39^\circ)} = \frac{a}{\sin(98^\circ)}$$

$\Rightarrow b = \dfrac{22 \cdot \sin(43^\circ)}{\sin(39^\circ)}$ $\qquad$ $\Rightarrow a = \dfrac{22 \cdot \sin(98^\circ)}{\sin(39^\circ)}$

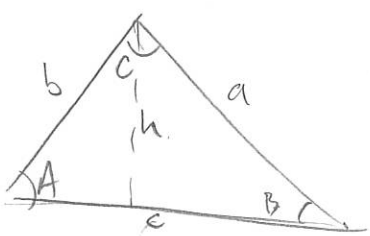
$\Rightarrow \boxed{b = 23.84 \text{ ft}}$ $\qquad$ $\Rightarrow \boxed{a = 34.62 \text{ ft}}$

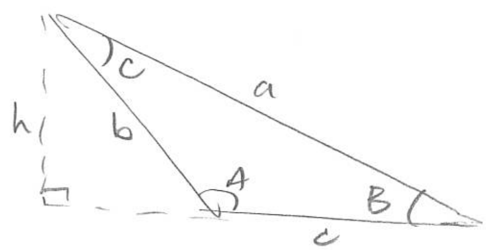
$\boxed{\text{WS}}$ #2 Use the Law of Sines for an ASA triangle

- The other case, when given SSA is a little more difficult. — This is known as the ambiguous case because three separate things can happen

$\boxed{\text{WB}}$ #2 - #5

- We can use the Law of Sines to find the area of an oblique triangle:





Area $= \frac{1}{2}(\text{base})(\text{height})$
$= \frac{1}{2} c \cdot h$
$= \frac{1}{2} c \cdot a \sin(B)$

$\sin(B) = \frac{h}{a}$
$\Rightarrow h = a\sin(B)$

Area $= \frac{1}{2}(\text{base})(\text{height})$
$= \frac{1}{2} c \cdot h$
$= \frac{1}{2} c \cdot a \sin(B)$

$\sin(B) = \frac{h}{a}$
$\Rightarrow h = a\sin(B)$

Similar arguments can be used to find the following general formulas.

Area of an oblique triangle:

$$\text{Area} = \frac{1}{2} ab\sin(C) = \frac{1}{2} ac\sin(B) = \frac{1}{2} bc\sin(A).$$

$\boxed{\text{WS}}$ #6 Find the area of an oblique triangle.

Section 3.2: Law of Cosines

- The other main theorem of this section is called the Law of Cosines which is used for oblique triangles of the form SSS or SAS.

- Law of Cosines: For any triangle ABC, we have

Standard form:	Alternative forms.
$a^2 = b^2 + c^2 - 2bc\cos(A)$	$\cos(A) = \dfrac{b^2+c^2-a^2}{2bc}$
$b^2 = a^2 + c^2 - 2ac\cos(B)$	$\cos(B) = \dfrac{a^2+c^2-b^2}{2ac}$
$c^2 = a^2 + b^2 - 2ab\cos(C)$	$\cos(C) = \dfrac{a^2+b^2-c^2}{2ab}$

$\boxed{\text{WS}}$ #1 Use Law of Cosines to find SSS triangle.

Note: when using the Law of Cosines, it is always beneficial to find the largest angle first, i.e. the angle opposite the longest side

$\boxed{\text{WS}}$ #2 Solve a SAS triangle.

- Using the Law of Cosines, we can prove the following formula for finding the area of any triangle:

- Heron's Area Formula: Given any triangle with sides a, b, c, the area is

$$\text{Area} = \sqrt{s(s-a)(s-b)(s-c)} \quad \text{where } s = \frac{a+b+c}{2}$$

$\boxed{\text{WS}}$ #3 Using Heron's Formula