CHAPTER 2 – Limits & the Derivative

Section 2.2: Limits All Around the Plane

- **Def**: Let $f(x)$ be defined for all values x near a value c, but not necessarily at c. We say that the limit of $f(x)$ is L as x approaches c, and we write

$$\lim_{x \to c} f(x) = L$$

if the value of $f(x)$ is closer and closer to L as x takes on values closer and closer (on both sides) to c.

Note: The value of the expression "$\lim_{x \to c} f(x)$", if it exists, only depends on values of f near c. It tells us nothing about when $x = c$ i.e. $\lim_{x \to c} f(x)$ does not have to be the same as $f(c)$!

- Easiest way to compute a limit is to examine a graph of $f(x)$ near the value c.

$\boxed{\text{WS}}$ #1 working with limits graphically

- When we write $\lim_{x \to c} f(x)$, we interpret this as a two-sided limit, i.e. x is approaching c from values smaller (left) and larger (right) than c.

- <u>Def</u>: Suppose f is defined for all x near c with $x<c$, then we write

$$\lim_{x \to c^-} f(x) = L$$

to mean that the function value approaches L as x approaches c from the left.

Likewise, we write the right sided limit as

$$\lim_{x \to c^+} f(x) = L.$$

- In general, for a one-sided limit to exist, the function must be approaching a single value L.
- In order for a two-sided limit to exist, the one-sided limits must exist <u>and</u> they must be equal:

$$\lim_{x \to c^-} f(x) = L = \lim_{x \to c^+} f(x) \text{ is equivalent to } \lim_{x \to c} f(x) = L$$

- So... when does a limit not exist? There are three typical cases:

1.) Function is unbounded near c

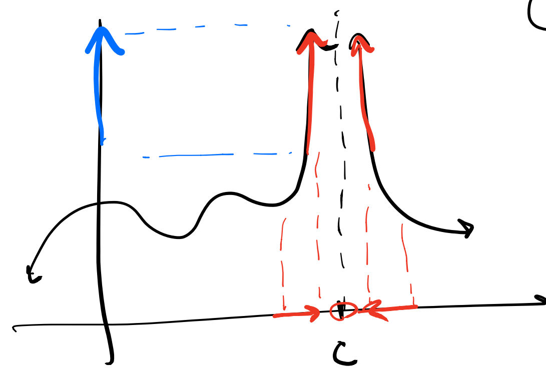
Infinite Limits

Typical of rational functions

$$\lim_{x\to c^-} f(x) = \infty$$ ← not a single value

$$\lim_{x\to c^+} f(x) = \infty$$

$$\lim_{x\to c} f(x) = \infty$$ ← Although we write ∞ to mean the function grows without bound, the limit technically does not exist.



2.) The one-sided limits are not the same.

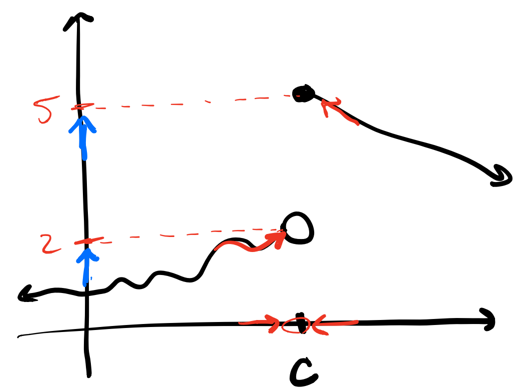
Jump Discontinuity

Typical of Piecewise functions

$$\lim_{x\to c^-} f(x) = 2$$

$$\lim_{x\to c^+} f(x) = 5$$

$$\lim_{x\to c} f(x) = \text{DNE}$$ ← Although the individual one-sided limits exists, the two-sided limit does not i.e. they don't line up.



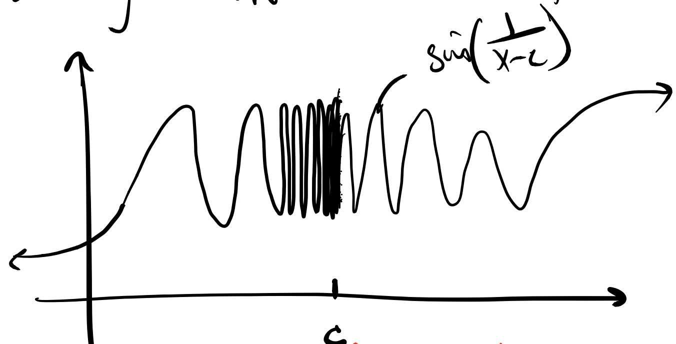
3.) Function doesn't settle to a single value i.e. oscillates.

Can occur with trig functions.

$$\lim_{x\to c^-} f(x) = \text{DNE}$$

$$\lim_{x\to c^+} f(x) = \text{DNE}$$

$$\lim_{x\to c} f(x) = \text{DNE}$$



In all cases, the function continues to oscillate as x approaches c. Never settles.

$\boxed{\text{WS}}$ #3 working more with graphical limits.

- We can provide a more precise definition of a vertical asymptote using the language of limits.
- Def: we say a function has a vertical asymptote at the value $x = c$ if any one of the following limits hold:

$$\lim_{x \to c^-} f(x) = \pm\infty, \quad \lim_{x \to c^+} f(x) = \pm\infty, \quad \lim_{x \to c} f(x) = \pm\infty$$

- we can also compute limits simply by numerically computing function values at points near c. I call this the "Brute Force" Method and it is essentially fail-safe.

$\boxed{\text{WS}}$ #4 Compute the limit numerically.

- we've encountered infinite limits, which occur when the function value grows without bound. We can also compute a limit at $\pm\infty$ by examining the function values as x values increase/decrease without bound.

$\boxed{\text{WS}}$ #1 Investigating a limit at infinity numerically.

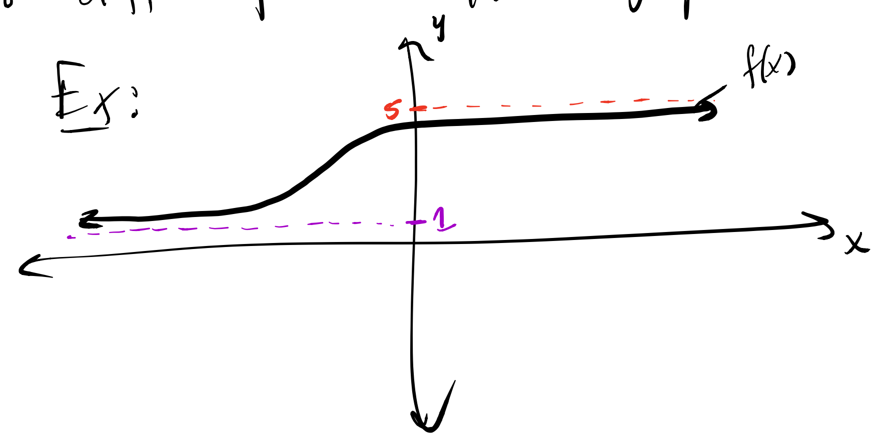
⑤

• **Def**: We say the line $y=L$ is a horizontal asymptote if either of the following limits hold:

$$\lim_{x\to-\infty} f(x)=L \qquad \lim_{x\to\infty} f(x)=L$$

• By definition, a function can have a maximum of two horizontal asymptotes. Is it possible for a function to have two different horizontal asymptotes?

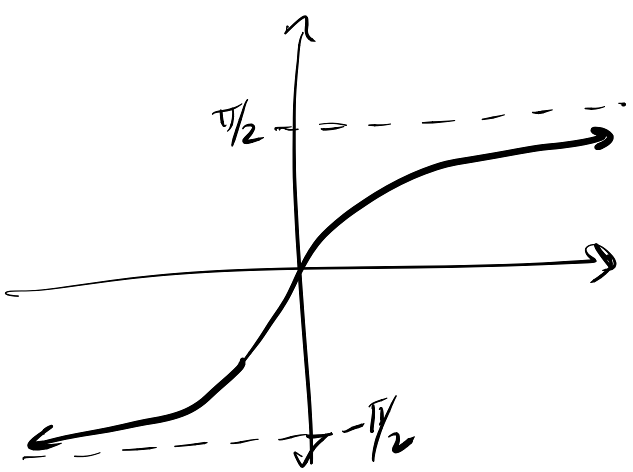
Ex:



$\lim_{x\to-\infty} f(x)=1$ (HA at $y=1$)

$\lim_{x\to\infty} f(x)=5$ (HA at $y=5$)

• Here's another example: Consider the function $f(x)=\arctan(x)$



$\lim_{x\to-\infty} \arctan(x)=-\frac{\pi}{2}$

↳ HA at $y=-\frac{\pi}{2}$

$\lim_{x\to\infty} \arctan(x)=\frac{\pi}{2}$

↳ HA at $y=\frac{\pi}{2}$.

• **Def**: The end behavior of a function is the behavior of the function as $x\to-\infty$ and as $x\to\infty$.

- Polynomials do not have horizontal asymptotes. The end behavior of a polynomial function is unboundedness — infinite limit at infinity.
- Exponential functions have a horizontal asymptote.

$\boxed{\text{WS}}$ #4–#5 End behavior of functions.

Section 2.4: Determining Limits Analytically

- Without the graph, there are plenty of techniques we can take advantage of to aid in evaluating limits.
- Basic Limit Laws: let f and g be two functions for which

$$\lim_{x\to c} f(x) \quad \text{and} \quad \lim_{x\to c} g(x)$$

both exist. Then

1.) Sum/Difference Law:

$$\lim_{x\to c}\left[f(x)\pm g(x)\right] = \lim_{x\to c} f(x) \pm \lim_{x\to c} g(x)$$

2.) Constant Mult. Law:

$$\lim_{x\to c}\left[k f(x)\right] = k \lim_{x\to c} f(x) \quad (k = \text{constant})$$

3.) Product Law:

$$\lim_{x\to c}\left[f(x) g(x)\right] = \lim_{x\to c} f(x) \cdot \lim_{x\to c} g(x)$$

4.) Quotient Law:

$$\lim_{x\to c}\left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}, \text{ provided } \lim_{x\to c} g(x) \neq 0.$$

5.) Rational Power Law: Let m and n be nonzero integers with no common factors, then

$$\lim_{x \to c} [f(x)]^{m/n} = \left[\lim_{x \to c} f(x)\right]^{m/n}$$

$\boxed{WS}$ #1–#2 Working with basic limit laws.

∘ Although evaluating a limit means not plugging in the value of interest, there are some functions for which you can plug in the value to evaluate the limit.

∘ Thm: (Polynomial Sub Law) Let $p(x)$ be a polynomial, then

$$\lim_{x \to c} p(x) = p(c).$$

∘ Thm (Rational Function Law) Let $\frac{p(x)}{q(x)}$ be a rational function where $q(c) \neq 0$, then

$$\lim_{x \to c} \frac{p(x)}{q(x)} = \frac{p(c)}{q(c)}$$

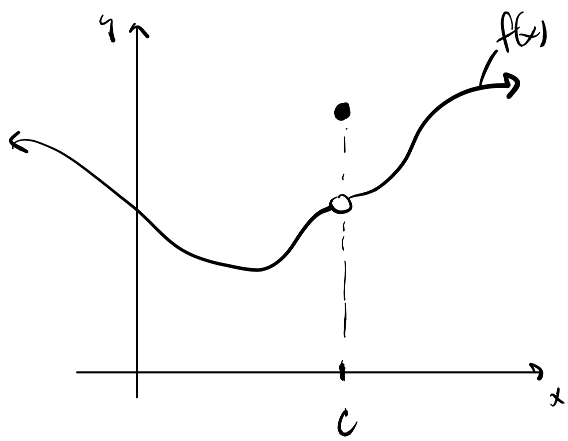
∘ The bottom line: if c is in the domain of the function, then evaluating the limit is equivalent to plugging in the value c into the function.

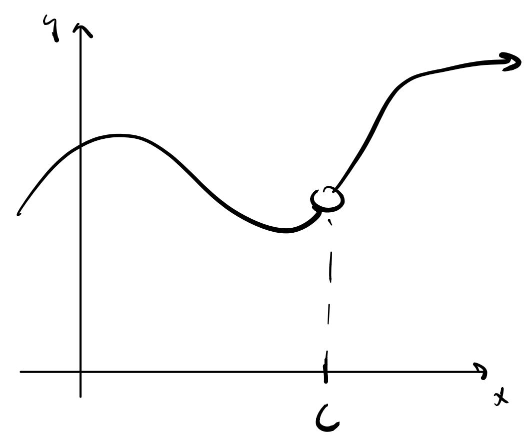
↳ Not true for piecewise functions!

$\boxed{WS}$ #3–#4 Working with polynomials & rational functions.

- There are two other techniques that can prove very useful:

1.) Removing the "bad" point: we can use algebra to "fix" the case when the limit $\lim_{x \to c} f(x)$ exists but the $\lim_{x \to c} f(x) \neq f(c)$. There are generally two cases:





- $\lim_{x \to c} f(x)$ exists
- $f(c)$ exists

Not Equal

- $\lim_{x \to c} f(x)$ exists
- $f(c) = \text{DNE}$

still not equal.

- In both cases, we can use algebra to evaluate the limit.

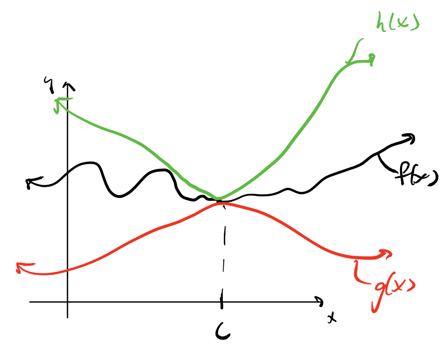
$\boxed{\text{WS}}$ #5a – #5b Removing the bad point.

2.) The Squeeze Theorem: If $g(x) \leq f(x) \leq h(x)$ for all x in an open interval containing c, except possibly at c, and if

$$\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L,$$

then

$$\lim_{x \to c} f(x) = L.$$



$\boxed{\text{WS}}$ #1 – #2 working with the Squeeze Thm.

- We must practice more limits, especially limits involving the difference quotient.

$\boxed{WS}$ #3a-3c working with difficult limits.

Section 2.5: Continuity.

- Def: A function is continuous at c if the limit value is equal to the function value, i.e.

$$\lim_{x \to c} f(x) = f(c).$$

Continuity Check list:

1.) f is defined at c (c has to be in domain)

2.) the limit exists $\left(\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) \right)$

3.) The limit is equation to the function value i.e. Part 1 is equal to Part 2 $\left(\lim_{x \to c} f(x) = f(c) \right)$

$\boxed{WS}$ #1, #2 working with continuity graphically and algebraically.

- A function is said to be continuous on its domain, if it is continuous at every point in its domain.
- Because of the polynomial and rational function substitution law for limits, it follows directly that polynomial and rational functions are continuous on their domains.

- **Def:** we define f to be right-continuous at c if

$$\lim_{x \to c^+} f(x) = f(c)$$

and left-continuous at c if

$$\lim_{x \to c^-} f(x) = f(c).$$

Note: this definition is especially useful if the value c is in the domain of f as a left or right endpoint.

$\boxed{\text{WS}}$ #1-#3 working with continuity and right/left continuity.

- As we seen in the worksheet examples, a function $f(x)$ can have a single discontinuity where the limit exists but it does not match up with the function value. We call this a removeable discontinuity.

- **Def:** If a function $f(x)$ has a discontinuity at c but the limit $\lim_{x \to c} f(x)$ exists, c is called a removable discontinuity of f.

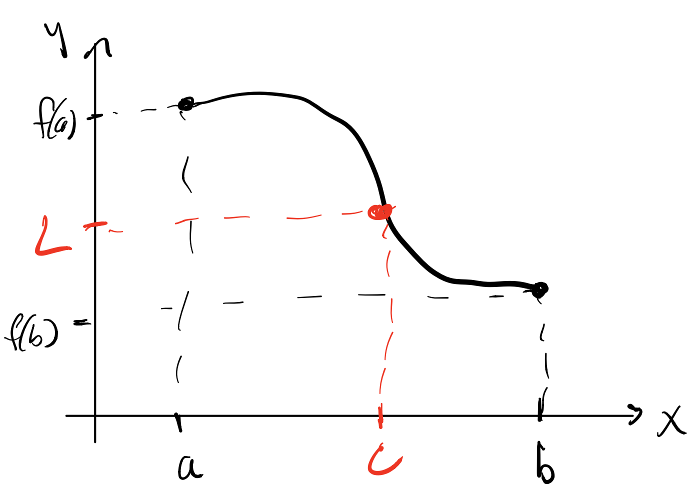
- **Note:** For any removable discontinuity, the function f can be made continuous by redefining the function value at c using a piecewise function

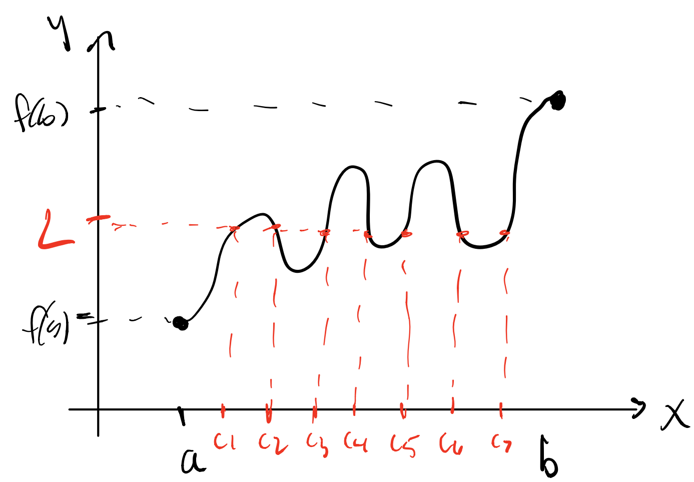
- We can also define a less strict version of continuity. Even though a function may not be continuous at a point c, it may be "right" or "left" continuous at the point c.

$\boxed{\text{WS}}$ #4 Working with removable discontinuities.

Thm: (Intermediate Value Theorem) If f is a continuous function defined on the closed interval $[a,b]$, then f takes on every value between $f(a)$ and $f(b)$. That is, if L is a real number such that $f(a) \le L \le f(b)$, then there exists a c inside (a,b) such that $f(c) = L$. (1)

• Graphically, this means





Since we must connect the points $(a, f(a))$ and $(b, f(b))$ through continuity, there must be a value c in (a,b) that corresponds to a function value L such that $f(c) = L$ where L is in between $f(a)$ and $f(b)$.

[WS] #5 working with an example of the IVT

Section 2.1: Rates of Change and Tangents

• Now that we have limits under control, we can start to study rates of change by looking at the classic velocity problem:

• In short, distance = rate × time or

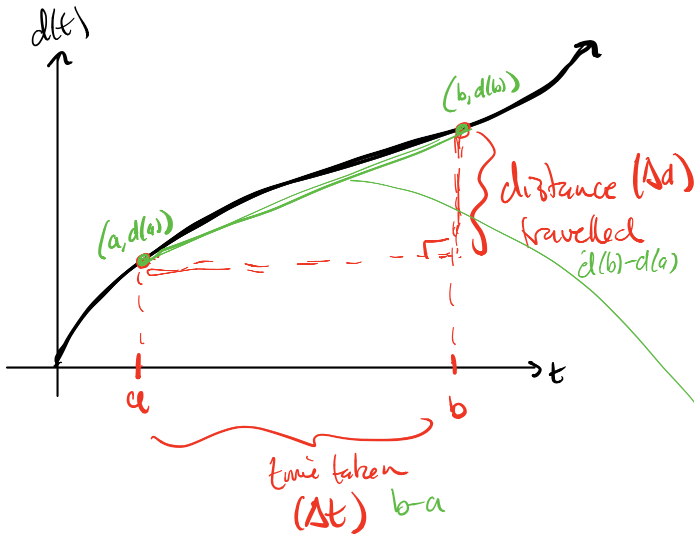
distance (miles) $\boxed{d = rt}$ time (hrs)

rate/velocity (miles/hr)

The rate r is taken to be the average rate of distance over a time span t.

(Not too realistic as many applications have varying speeds over miniscule time intervals.)

- By considering distance function where $d(t)$ gives the distance traveled after t time units, we can construct an approximation for the velocity using $r = \frac{d}{t}$:



- Although the velocity is changing over a to b, we can estimate it by finding the average velocity:

$$[\text{Avg velocity}] = \frac{\text{distance travelled}}{\text{time taken}} = \frac{\Delta d}{\Delta t}.$$

Slope of secant line $= \frac{\Delta d}{\Delta t} = \frac{d(b)-d(a)}{b-a}.$

- The average velocity over the time interval $[a,b]$ is the slope of the <u>secant line</u> connecting the points $(a, d(a))$ and $(b, d(b))$.

in other words

$$[\text{Avg velocity}] = \frac{\Delta d}{\Delta t} = \frac{d(b)-d(a)}{b-a}.$$

called the "Difference Quotient"

$\boxed{\text{WS}}$ #1 working with finding the average velocity.

- we can construct the slope of the secant line for any function over an interval $[a,b]$. In the general sense, the difference quotient is interpreted as a rate of change.

- Applying this idea to any function $f(x)$: Consider the function f over any interval $[a,b]$, then

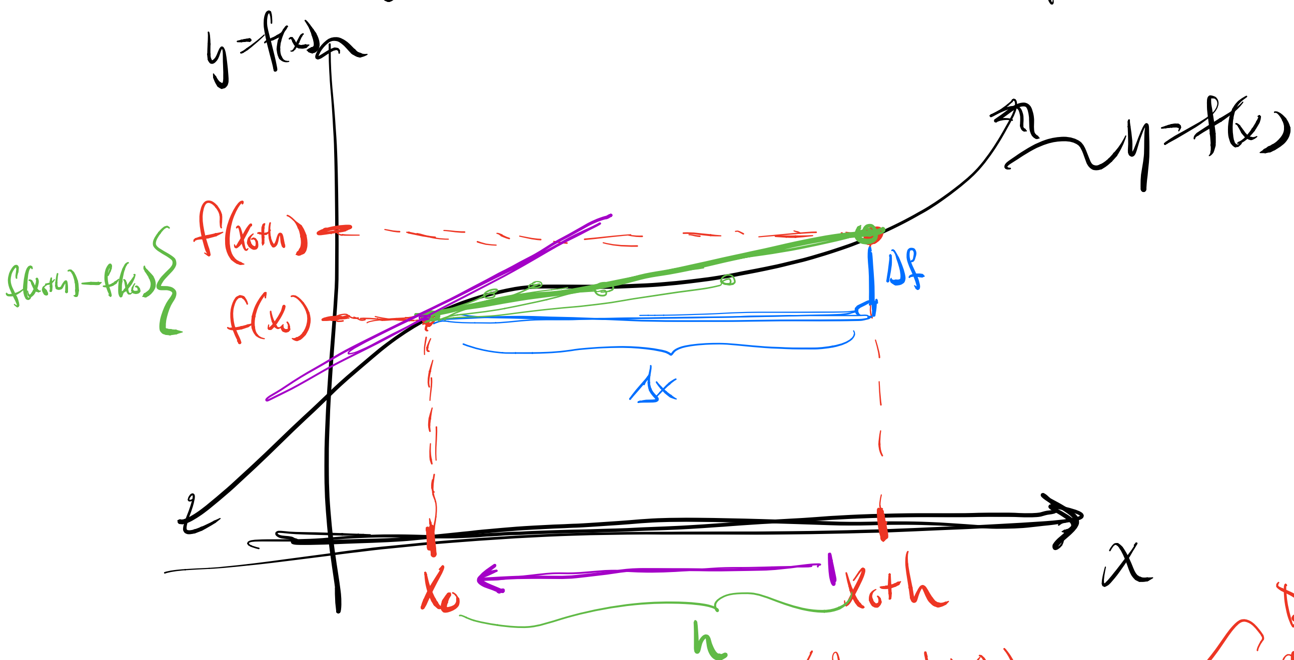
- Δx = change in x values
- Δf = change in function values
- $\frac{\Delta f}{\Delta x}$ = average rate of change of f over $[a,b]$.

- For example if $f(x)$ is the # of covid-19 infections at time x, then $\frac{\Delta f}{\Delta x}$ is the average infectious rate:

$$\boxed{[\text{Avg Rate of Change}] = \frac{\Delta f}{\Delta x} = \frac{f(b)-f(a)}{b-a}}$$

Difference Quotient over $[a,b]$

- Another way to represent the difference quotient:

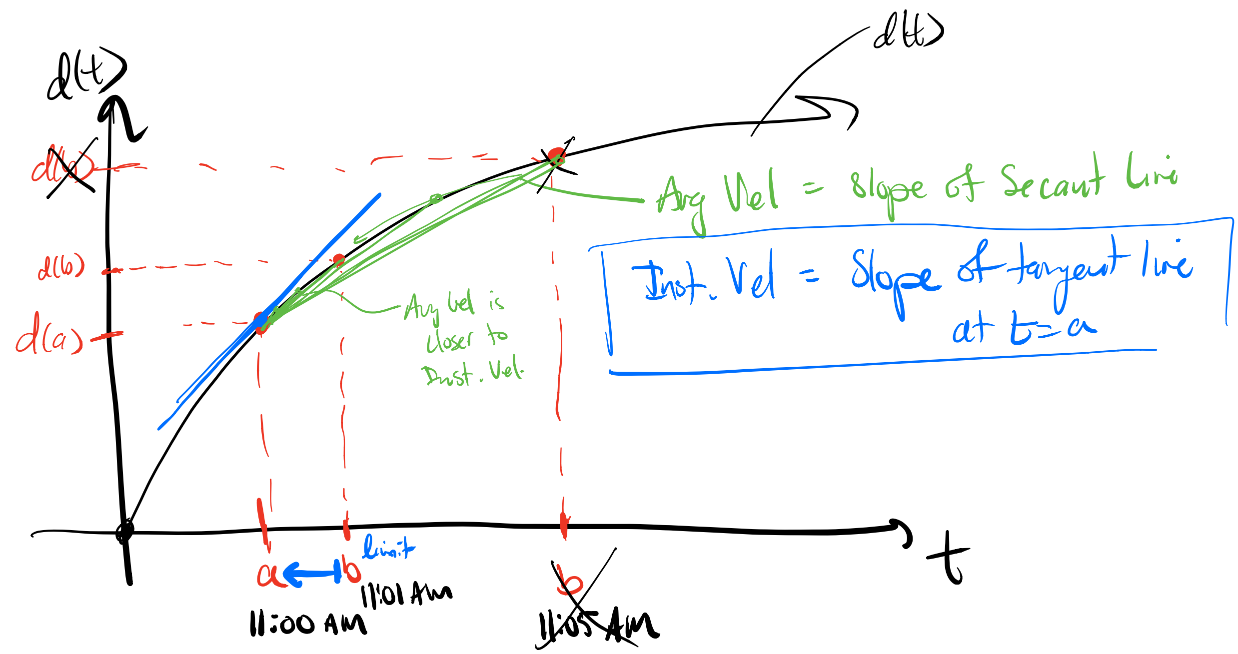


Difference quotient at x_0 with increment h.

$$\boxed{[\text{Avg Rate of Change}] = \frac{\Delta f}{\Delta x} = \frac{f(x_0+h)-f(x_0)}{h}}$$

might use this more

• How might we calculate the instantaneous velocity at a single point, say $t=a$? The average velocity must be computed over some finite time interval i.e. $[a,b]$. But the instantaneous velocity occurs at a single point i.e. over a time interval of length 0! Consider a distance function $d(t)$:



• To obtain the instantaneous velocity at time $t=a$, we would continually compute the average rate of change but for smaller and smaller time intervals i.e. we let b get infinitesimally closer to a! This is a limit!

$$\boxed{\underbrace{\left[\text{Inst. velocity at } a\right]}_{\text{slope of tangent line}} = \lim_{b\to a} \frac{d(b)-d(a)}{b-a}}$$

Avg velocity

this limit makes it the instantaneous velocity.

• Just as we did before, we can adapt this idea to any function $f(x)$ and we can formulate it in two different ways:

$$\left[\text{Inst. Rate of Change at } a\right] = \lim_{b \to a} \frac{f(b)-f(a)}{b-a}$$

$$\left[\text{Inst. Rate of Change at } x_0\right] = \lim_{h \to 0} \frac{f(x_0+h)-f(x_0)}{h}$$

- Geometrically, the number that results from this limit is called the slope of the tangent line to the function $f(x)$ at that point.

So... in short

Avg Rate of Change $\Leftrightarrow$ Slope of Secant Line $\Leftrightarrow$ $\frac{f(b)-f(a)}{b-a}$

Inst Rate of Change $\Leftrightarrow$ Slope of Tangent Line $\Leftrightarrow$ $\lim_{b \to a} \frac{f(b)-f(a)}{b-a}$.

- The slope of the Tangent Line at a point is just called the rate of change of f at that point.

$\boxed{\text{WS}}$ #2 Working with Tangent Lines

- Since the instantaneous rate of change is nothing more than a limit, we can compute it using the Brute Force Method.

$\boxed{\text{WS}}$ #3 Compute the instantaneous velocity of a function.

Section 2.6: The derivative

- The slope of the tangent line (i.e. the inst. velocity) is so important, that we give it a special name.
- Def: The derivative of the function f at the point c, denoted by $f'(c)$, is

note: we replaced x_0 with c.

$$f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

provided the limit exists. (pronounced "f prime")

$\boxed{\text{WS}}$ #1 – ~~#2~~ working with the derivative.

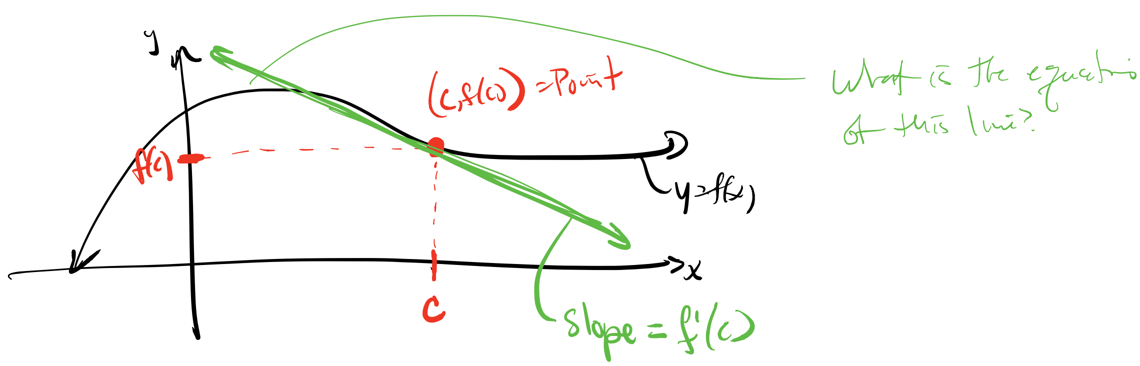
- The result of this limit has many interpretations:
  - $f'(c)$ is the derivative of f at the point $x=c$.
  - $f'(c)$ is the slope of the tangent line that intersects the graph of f at the point $(c, f(c))$.
  - $f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$
  - $f'(c)$ is the instantaneous rate of change of f at c.

- In some scenarios, it may be beneficial to use the following alternative formula when evaluating the derivative:

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

Note: $a=c$ and $b=x$ in the previous definition.

• Now, since $f'(c)$ represents the slope of the tangent line, and this tangent line intersects the graph of f at $(c, f(c))$, we can derive a formula for the Equation of the tangent line.



• We can use the point-slope form of a line to construct the equation of the tangent line:

$$\text{Point-Slope Form: } y - y_1 = m(x - x_1)$$

$$(x_1, y_1) = (c, f(c))$$

$$m = f'(c)$$

$$\Rightarrow \boxed{y - f(c) = f'(c)(x - c)}$$

Point-Slope Form of Tangent Line

$$\Rightarrow y = f'(c)x - f'(c)c + f(c)$$

$$\Rightarrow \boxed{y = \underbrace{f'(c)}_{m}x + \underbrace{(f(c) - f'(c)c)}_{b}}$$

y-int Form of Tangent Line

• Therefore, we have the following result: The equation of the tangent line to the function $y = f(x)$ at the point $(c, f(c))$ is

<u>Point-Slope</u>

$$y - f(c) = f'(c)(x - c)$$

<u>y-int</u>

$$y = f'(c)x + (f(c) - cf'(c))$$

$\boxed{\text{WS}}$ #3 - #4 Finding tangent lines.

- we've been finding the slope of the tangent line (i.e. $f'(c)$) for any given point $x=c$ in the domain of f. Instead of finding the slope at one point at a time, we can assume x is any point in the domain of f and find the "derivative" function associated to f.

- **Def:** Let x be in the domain of f. Then the derivative function, denoted by $f'(x)$, is the function

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

we changed c to x in the previous definition

This function has the domain consisting of all x values for which the above limit exists.

Note: Given $f(x)$, we know $y=f(x)$ represents a curve in the x-y plane. This function $f'(x)$ takes in the x-coordinate and outputs the slope of f at x.

$\boxed{\text{WS}}$ #1–#3 Deriving the derivative function.