CHAPTER 1 - A Function Primer

Section 1.1: Functions

- Before we get started on functions, let's talk about basic algebra techniques and exponent laws.

$\boxed{\text{WS}}$ #1, #2 Working with fractions and exponents.

- **Def**: For any two sets A and B, a <u>relation</u> from A to B is a set of ordered pairs (a,b), where $a \in A$ and $b \in B$.

  - A relation is typically represented by a two variable equation in x and y (x, y represent real #s)

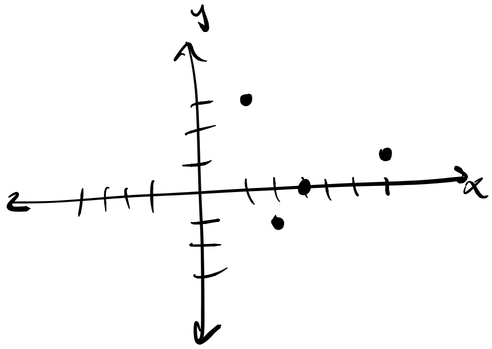
    Ex: $y = 2x + 3$ ← relation, looks like a line on the Cartesian plane.

    Ex: $x^2 + y^2 = 4$ ← relation, looks like a circle centered at $(0,0)$ with radius 2.

- In this sense, a relation is the set of all ordered pairs (x,y) that <u>satisfy</u> the equation.

- A relation can be a finite set of points:

$$R = \{(1,3),\ (2,-1),\ (3,0),\ (6,1)\}$$



Domain of R: $\{1,2,3,6\}$

Range of R: $\{-1,0,1,3\}$

- Def: A function f from the set A to a set B is a special relation with the additional property that each element $a \in A$ is related to exactly one element $b \in B$.

  - we write, $b = f(a)$ or $f(a) = b$ to mean this unique element that f assigns to a, i.e.

    "b is the value of f at a"

  - we may also write,

$$f : A \longrightarrow B$$

  to mean f is a function that "maps" elements from the domain A to the codomain (range) B.

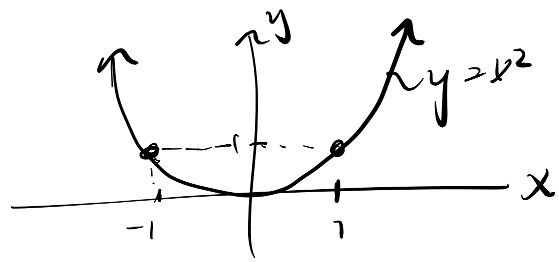
- The elements of the domain make up the independent variable, typically denoted by x.
- The assignments of all x in the domain, say $f(x)$, make up the dependent variable, typically called y, in the range. i.e. $y = f(x)$

- The domain of any function is usually implied i.e. the set of all real #'s for which a real # is the output when we apply the function f.
- There's really only three operations we have to watch out for:
  1.) Dividing by zero
  2.) Taking the even root of a negative #
  3.) Taking the logarithm of a non-positive #

$\boxed{\text{WS}}$ #3–#5, working with domains.

- The best ways to describe a function:
  1.) mathematical formula Ex: $f(x) = x^2$
  2.) Graph on Cartesian Plane Ex: $y = x^2$

x	$y = x^2$	(x, y)
-2	4	$(-2, 4)$
-1	1	$(-1, 1)$
0	0	$(0, 0)$
1	1	$(1, 1)$
2	4	$(2, 4)$



- Using the graph, we can tell several things about a relation
  1.) The Vertical Line Test – the relation is a function if no vertical line passes through the graph more than once.
  2.) Increasing/Decreasing – as we read the graph from left to right, are the function values increasing or decreasing?

- increasing: $x_1 < x_2$ implies $f(x_1) \le f(x_2)$
- decreasing: $x_1 < x_2$ implies $f(x_1) \ge f(x_2)$
- monotone: f is strictly inc or strictly dec over an interval.

3.) Odd/Even – symmetry of a function is easily observed by the graph.

- odd: f is odd if $f(-x) = -f(x)$ for all x in domain (origin symmetric)
- even: f is even if $f(-x) = f(x)$ for all x in domain (y-axis symmetric)

4.) Domain/Range – each set is easily deduced by the graph

- domain: the portion of the x-axis on which the shadow of the graph is cast
- range: the portion of the y-axis on which the shadow of the graph is cast.

$\boxed{\text{WS}}$ #6–#8 work with graphs and symmetry of functions.

Section 1.2: A Function Repertory

- It is very important to know the types of functions that we'll be dealing with in this class.

- **Algebraic Functions**: functions created using addition, subtraction, multiplication, division, or exponents.

- Includes the following:

1.) **Polynomials**: $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$

coefficients

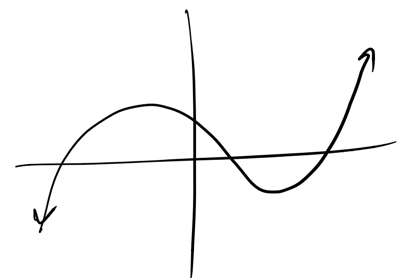
degree (leading/highest power)

Ex: $f(x) = x^2 + 3x - 1$

$f(x) = 4x^{20} + 9x^{18} - 13x - 4$

$f(x) = 3x + 4$

Typical Graph:



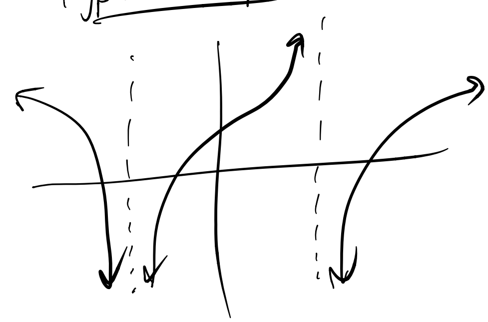
2.) **Rational Functions**: $f(x) = \dfrac{p(x)}{q(x)}$, where $p(x)$ & $q(x)$ are polynomials

Ex: $f(x) = \dfrac{x^2+1}{x^2+2x-15}$

$f(x) = \dfrac{1}{3x+5}$

$f(x) = \dfrac{x^{100}+4x-3}{x^{98}-3x^{47}+2}$

Typical Graph:

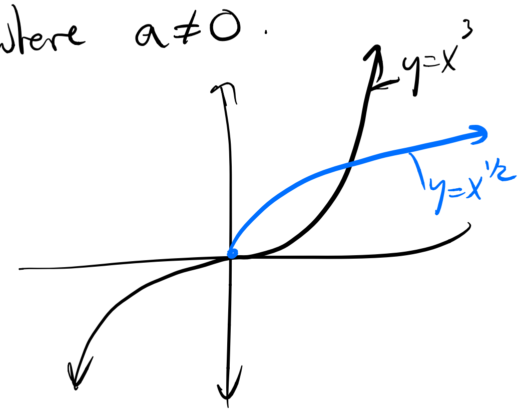


3.) **Power Functions**: $f(x) = x^a$, where $a \neq 0$.

Ex: $f(x) = x^2$

$f(x) = x^{1/2} = \sqrt{x}$

$f(x) = x^{1/3} = \sqrt[3]{x}$



4.) Miscellaneous / other combinations of the above three types of functions:

Ex: $f(x) = -4x^6 + 7x^4 - \sqrt[3]{2x^3 - 1}$

$f(x) = \frac{3}{x} + 4x^{-1/100} - 3x^2 - \frac{4}{\sqrt{x+1}}$

- Transcendental Functions: class of functions that are not algebraic (transcends elementary algebra)

1.) Trigonometric Functions: the six trig functions

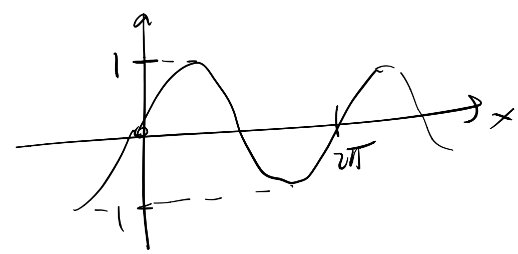
Ex: $f(x) = \sin(x)$

$f(x) = \cos(x)$

$f(x) = \tan(x)$

$f(x) = \csc(x)$

Typical Graph:



2.) Exponential Functions: for $a \neq 1$, $f(x) = a^x$

(base is constant, variable in the exponent)

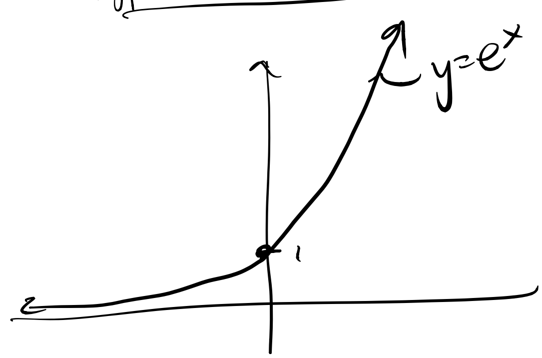
Ex: $f(x) = 2^x$

$f(x) = e^x$

$f(x) = \left(\frac{1}{2}\right)^x$

$f(x) = 3^{-x}$

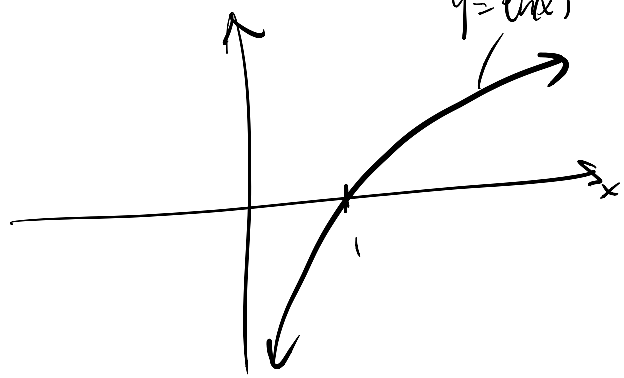
Typical Graph:



3) Logarithmic Functions: $f(x) = \log_a(x)$

Ex: $f(x) = \log_2(x)$
$f(x) = \ln(x)$

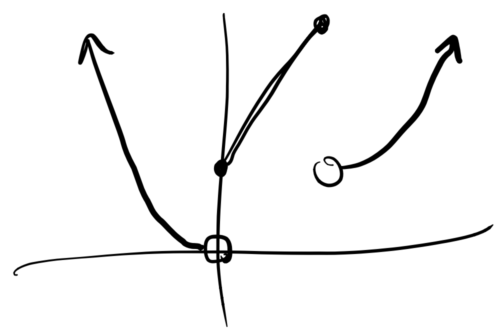
Typical Graph:



• Piecewise Defined Functions: a function defined differently on various parts of its domain.

Typical Graph:

Ex: $f(x) = \begin{cases} x^2 & \text{if } x<0 \\ 3x+1 & \text{if } 0 \le x \le 1 \\ 7-\sqrt{x+2} & \text{if } x>1 \end{cases}$



$f(x) = |x| = \begin{cases} -x & \text{if } x<0 \\ x & \text{if } x \ge 0 \end{cases}$

$\boxed{\text{WS}}$ #1–#4 working with functions

Section 1.3: Transformations & Combining Functions

· Recall that for any positive constant c, we can apply horizontal & vertical shifts to the graph of $y=f(x)$:

outside the argument

1.) $y=f(x)+c$ shifts $y=f(x)$ up c units.
2.) $y=f(x)-c$ shifts $y=f(x)$ down c units.

inside the argument

3.) $y=f(x+c)$ shifts $y=f(x)$ left c units.
4.) $y=f(x-c)$ shifts $y=f(x)$ right c units.

$\boxed{\text{WS}}$ #1 working with shifts.

- Recall that for a constant $k>1$, we can apply compression or stretching of the graph $y=f(x)$.

  1.) $y=cf(x)$ stretches $y=f(x)$ vertically by a factor of c
  2.) $y=\frac{1}{c}f(x)$ compresses $y=f(x)$ vertically by a factor of c
  3.) $y=f(cx)$ compresses $y=f(x)$ horizontally by a factor of c
  4.) $y=f(\frac{1}{c}x)$ stretches $y=f(x)$ horizontally by a factor of c

- Recall that we can reflect the graph of $y=f(x)$ across axes by multiplying ~~be~~ a negative:

  1.) $-f(x)$ reflects $f(x)$ by x-axis.
  2.) $f(-x)$ reflects $f(x)$ by y-axis.

$\boxed{\text{WS}}$ #2 working with graph transformations.

- Combining two or more functions to create new functions. Let $f(x)$ & $g(x)$ be given, then

  - $(f+g)(x) = f(x)+g(x)$
  - $(f-g)(x) = f(x)-g(x)$
  - $(fg)(x) = f(x)\cdot g(x)$
  - $(f/g)(x) = \dfrac{f(x)}{g(x)}$, $g(x)\neq 0$.
  - $(f\circ g)(x) = f(g(x))$ ← Function Composition "f composed of g"

$\boxed{\text{WS}}$ #3 – #4 working with combining functions.

- In general, we see that $(f\circ g)(x)$ is not the same as $(g\circ f)(x)$, i.e. order matters. In fact, we must be careful when computing the domain of a composite function.

$\boxed{\text{WS}}$ #5 – #6 working with composite functions.

Section 1.4: Inverse Functions

- Some functions may have an inverse function, which is another function that "undoes" the action of the original.
- However, this relationship may only be defined on a restricted domain.

Ex: Let $f(x) = x^3$, then the inverse function is denoted $f^{-1}(x) = x^{1/3}$.

Note: $f^{-1}(x) \neq \frac{1}{f(x)}$. $f^{-1}(x)$ is pronounced "f inverse"

- The defining characteristic of a function and its inverse is the following property:

$$f(f^{-1}(x)) = x \quad \text{and} \quad f^{-1}(f(x)) = x$$

That is, when you compose a function with its inverse (or vice versa), nothing acts on the value x.

Ex: $f(x) = x^3$ and $f^{-1}(x) = x^{1/3}$

- $f(f^{-1}(x)) = f(x^{1/3}) = (x^{1/3})^3 = x$
- $f^{-1}(f(x)) = f^{-1}(x^3) = (x^3)^{1/3} = x$.

✓ They are indeed inverses!

<u>Def</u>: Let f be <u>one-to-one</u> (i.e. f passes the horizontal line test), then f has an <u>inverse function</u>, denoted by f^{-1}.

- To find an inverse of $y=f(x)$, switch the roles of x and y, and solve the resulting eqn for y. What remains will be $y=f^{-1}(x)$.

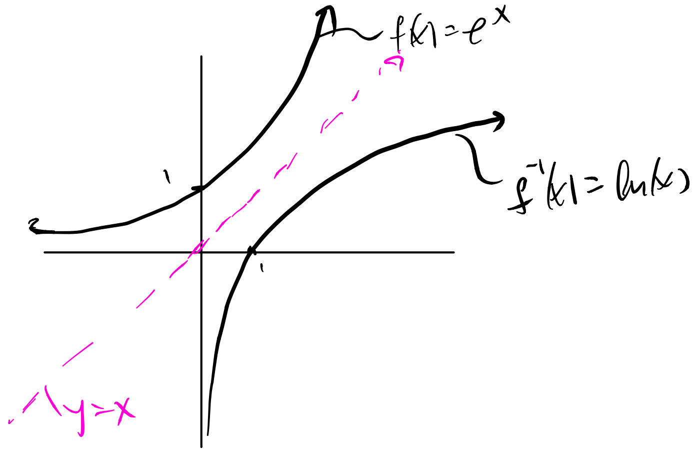
- Graphically, inverse functions have a nice relationship.
  - Domain of f is the Range of f^{-1}.
  - Range of f is the Domain of f^{-1}.
  - The graph of f^{-1} is the graph of f reflected about the line $y=x$.
  - If (a,b) lies on $y=f(x)$, then (b,a) lies on $y=f^{-1}(x)$ i.e.

    "if $f(a)=b$, then $f^{-1}(b)=a$."

$\boxed{\text{WS}}$ #1 working with inverse functions and their graphs.

- The <u>exponential</u> function has an inverse on its entire domain, we call it the <u>logarithm</u> function.

Ex: The function $f(x)=e^x$ is the natural exponential function. Its inverse is the natural logarithm function, $f^{-1}(x)=\ln(x)$.



- Since $f(0)=e^0=1$, we have $f^{-1}(1)=\ln(1)=0$.

- For any exponential function, $f(x)=b^x$, the inverse is $f^{-1}(x)=\log_b(x)$.

$$\Rightarrow \log_b(b^x)=x \quad \text{and} \quad e^{\log_b(x)}=x.$$

$\boxed{\text{WS}}$ #2 working with logarithms.

- Logarithms are used extensively to simplify complicated expressions involving exponents. This is done using the logarithm laws:

  - $\log_b(xy) = \log_b(x)+\log_b(y)$
  - $\log_b(x/y) = \log_b(x)-\log_b(y)$
  - $\log_b(x^y) = y\log_b(x)$

$\boxed{\text{WS}}$ #3–~~#5~~ #5 working with logarithms and solving equations.

- We can also define the inverse trig functions by restricting the domain:

$$f(x) = \sin(x) \longrightarrow f^{-1}(x) = \arcsin(x) \quad \text{on } [-\pi/2, \pi/2]$$

$$f(x) = \cos(x) \longrightarrow f^{-1}(x) = \arccos(x) \quad \text{on } [0, \pi]$$

$$f(x) = \tan(x) \longrightarrow f^{-1}(x) = \arctan(x) \quad \text{on } (-\pi/2, \pi/2)$$

$\boxed{\text{WS}}$ #6–#9 working with trig and inverse trig.