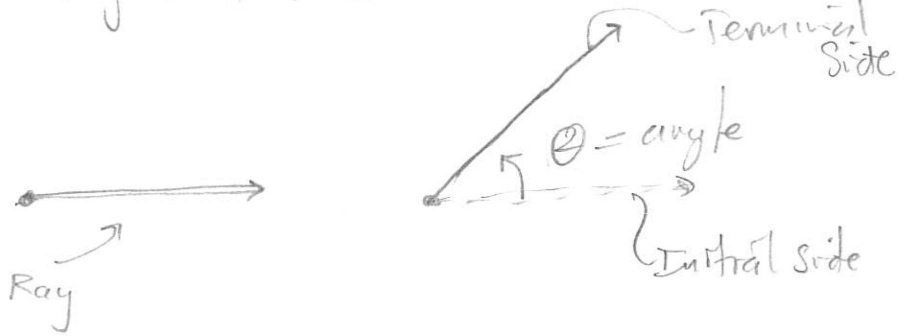
MAT 106 - Trigonometry
Brooks Emmerich

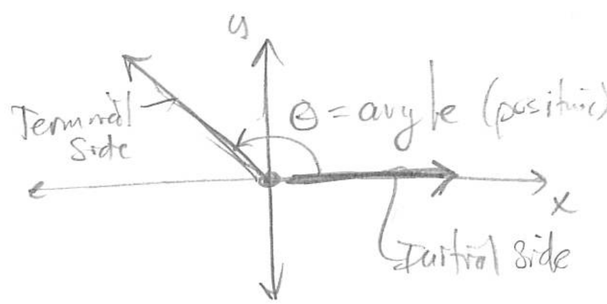
CHAPTER 1: Trigonometry

Section 1.1: Radian & Degree Measure

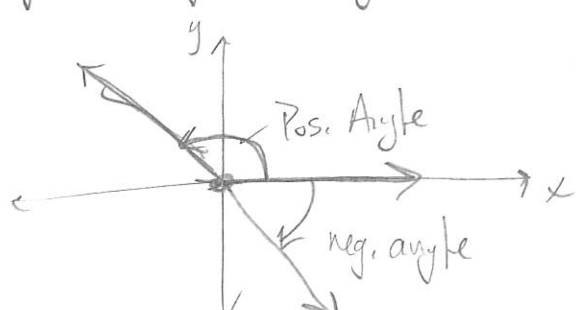
- Trigonometry = "Measurement of Triangles"
- Rotating a ray (half-line) about its vertex creates an angle.



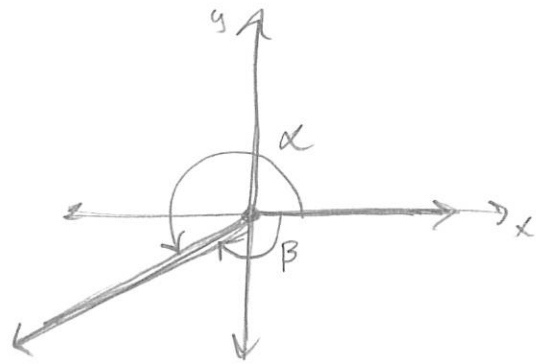
- we overlay the initial side on the positive x-axis and rotate the ray counterclockwise to create an angle on Cartesian plane.



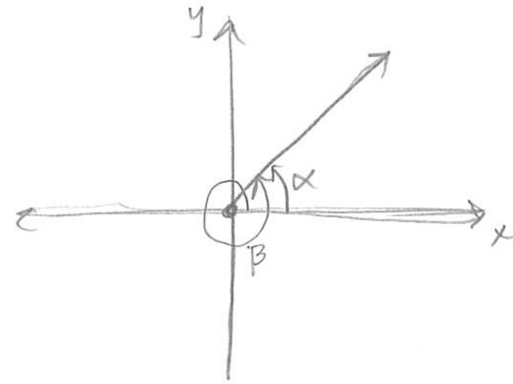
- Rotating clockwise yields negative angles.



- Two angles α and β that have identical terminal sides are called coterminal!

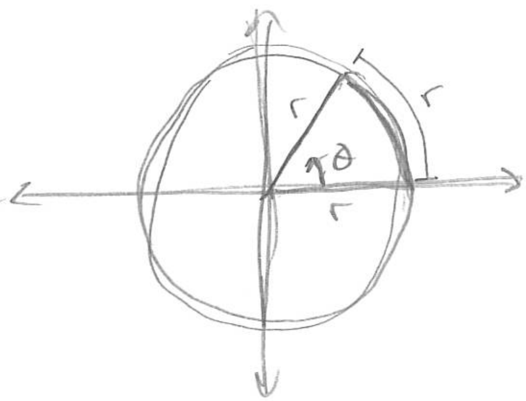


α, β are coterminal
($\alpha > 0$, $\beta < 0$)



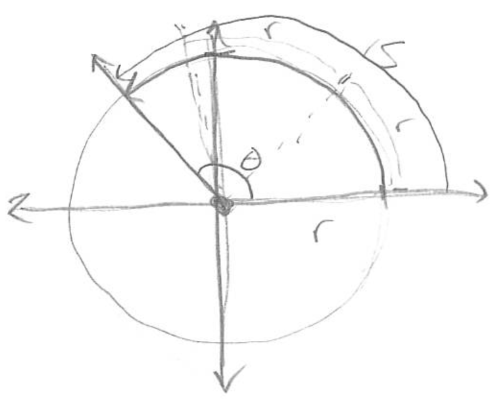
α, β are coterminal
($\alpha > 0$, $\beta > 0$)

- We need a measurement unit to determine the magnitude of an angle. The most widely used measure is the <u>radian</u>.
- Consider any circle of radius r:



- "$\theta = 1$ radian" means "the length of the arc created by the angle is exactly r units."

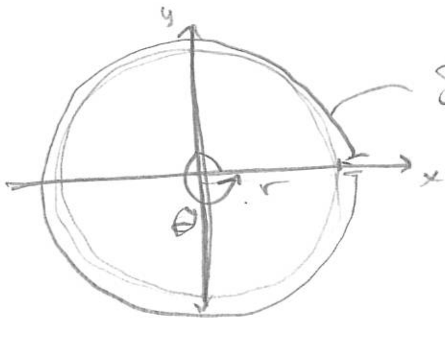
- In general, for any arc of length, say s, the central angle that creates this arc is exactly $\theta = \frac{s}{r}$ radians



$$S = \theta \cdot r$$

$$\Rightarrow \boxed{\theta = \frac{s}{r}} \text{ radians}$$

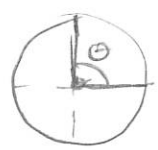
- Now, consider one <u>full</u> rotation of a circle of radius r.



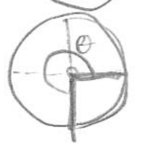
$$\theta = \frac{s}{r} = \frac{2\pi r}{r} = 2\pi$$

- Hence there are 2π radians in one full revolution!

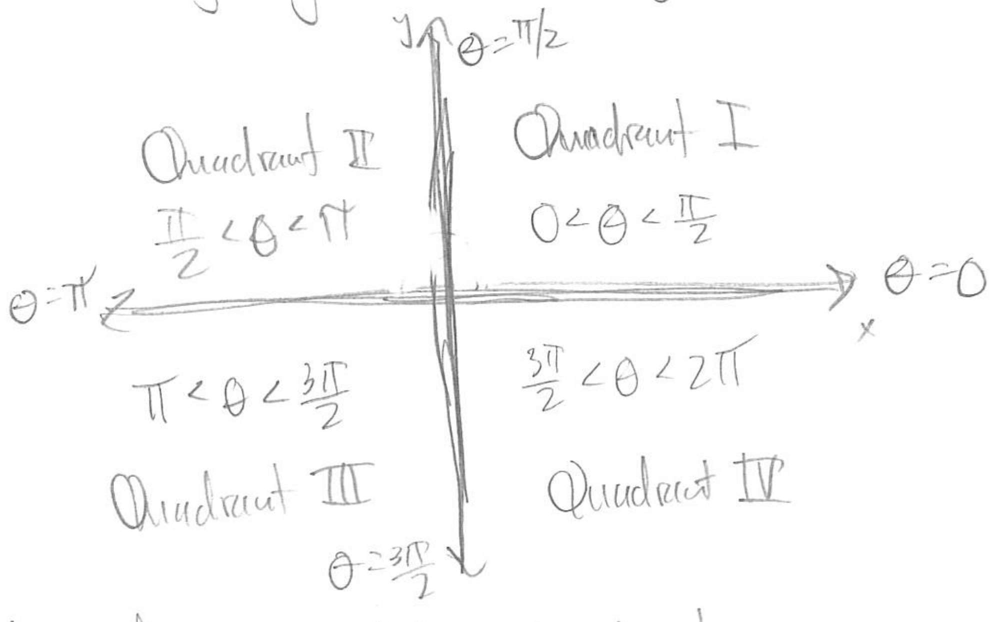
- This leads to the following:

<u>One-Fourth Revolution</u>:  $\theta = \frac{1}{4}(2\pi) = \frac{\pi}{2}$ rads

<u>Half Revolution</u>:  $\theta = \frac{1}{2}(2\pi) = \pi$ rads

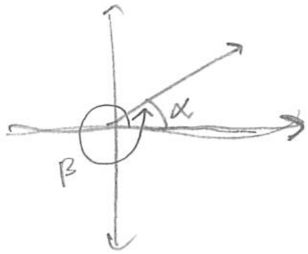
<u>Three-Fourth Revolution</u>:  $\theta = \frac{3}{4}(2\pi) = \frac{3\pi}{2}$

- Based on this, we can determine the quadrants of the Cartesian plane using angles between 0 and 2π radians.

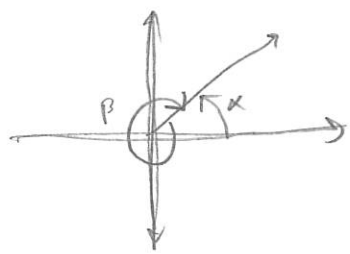


$\boxed{\text{WS}}$ #1 and working on identifying Quadrants.

- We can use the radian measure to calculate coterminal angles.



or

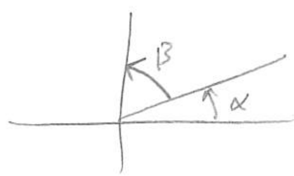


$\beta = \alpha + 2\pi$ or $\beta = \alpha - 2\pi$

$\boxed{\text{WS}}$ #2 a,b Finding coterminal angles.

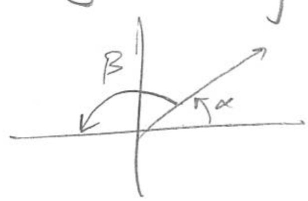
- Two angles α and β are complementary if they sum up to 90° or $\pi/2$ radians

$$\alpha + \beta = \pi/2$$



- Two angles α and β are supplementary if they sum up to 180° or π radians.

$$\alpha + \beta = \pi$$



$\boxed{\text{WS}}$ #3 a,b Finding complementary and supplementary angles.

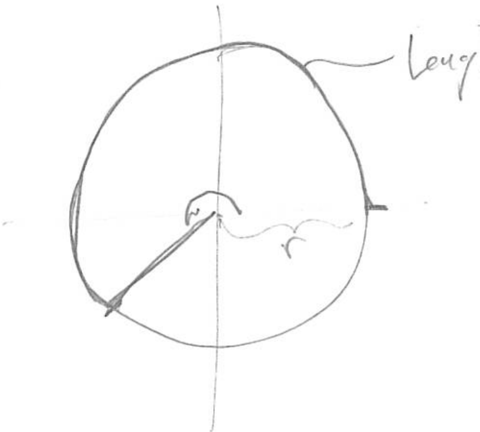
- Degrees is another angular measure that is popular, but we will typically work in radian measure. π radians $= 180^\circ$.

  1.) To convert from radians to degrees, multiply by $\frac{180^\circ}{\pi}$

  2.) To convert from degrees to radians, multiply by $\frac{\pi}{180^\circ}$

$\boxed{\text{WS}}$ #4, #5 a,b. Conversion of radians to degrees.

- One well known property of the radian measure is the arc-length of a piece of a circle:



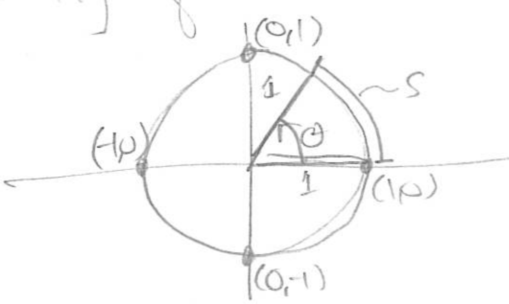
$$\boxed{s = r\theta}$$

r = radius of circle
θ : central angle (in radians!)

WS #6 Finding the length of an arc.

Section 1.2: The Unit Circle

- Consider the equation $x^2+y^2=1$. This describes the Unit Circle – the circle centered at $(0,0)$ with radius $r=1$.
- Since $r=1$, the arc-length of any arc on this circle has length exactly equal to $s=\theta$.

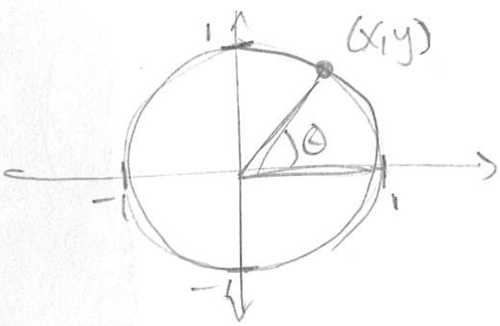


$$s = r\theta \Rightarrow s = (1)(\theta) \Rightarrow s = \theta\ !$$

- There are 16 special points on the unit circle that we will reference constantly. Each point corresponds to a specific angle.

WS #7 Showing that special points lie on unit circle.

- For each point on the unit circle, an angle is associated. For each angle, we define the x and y coordinate as the cosine and sine of that angle



- Each point (x,y) on Unit Circle corresponds to an angle θ
- We define cosine and sine as

$$x = \cos(\theta), \quad y = \sin(\theta)$$

- x-coord ⟺ cosine
  y-coord ⟺ sine

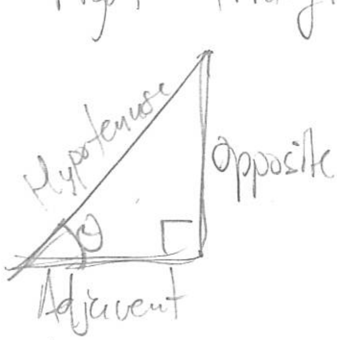
$\boxed{\text{WS}}$ #8 Finding the cosine and sine of specific angles.

$\boxed{\text{WS}}$ #1 Fill out the Unit Circle.

- Memorizing the Unit Circle allows us to know the cosine and sine of 16 specific angles without using a calculator.

Section 1.3: Right Triangle Trigonometry.

- We can use the Unit Circle as a guide for evaluating cosine and sine of any angle, but we can also define trig functions more generally for any right triangle:



- Here θ lies in the first quadrant.
- We define each of the six trig functions as simply ratios of these lengths

Trigonometry Lecture Notes

Cosine: $\cos(\theta) = \dfrac{Adj}{Hyp}$

Sine: $\sin(\theta) = \dfrac{Opp}{Hyp}$

tangent: $\tan(\theta) = \dfrac{\sin(\theta)}{\cos(\theta)} = \dfrac{Opp}{Adj}$

Reciprocal functions:

Secant: $\sec(\theta) = \dfrac{1}{\cos(\theta)} = \dfrac{Hyp}{Adj}$

Cosecant: $\csc(\theta) = \dfrac{1}{\sin(\theta)} = \dfrac{Hyp}{Opp}$

cotangent: $\cot(\theta) = \dfrac{1}{\tan(\theta)} = \dfrac{\cos(\theta)}{\sin(\theta)} = \dfrac{Adj}{Opp}$

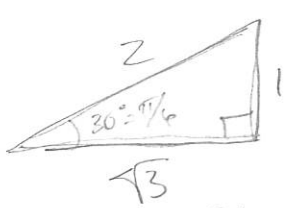
*Note: the last four functions are defined in terms of cosine and sine.

- Note: the functions simply reference ratios on a right triangle.

$\boxed{WS}$ # 1, 2 Evaluating the six trig functions.

There exist 3 prototype triangles that will help us to evaluate specific trig functions at certain angles. These triangles correspond to the Unit Circle

$\pi/6$ (30°)



$\cos(\pi/6) = \sqrt{3}/2$

$\sin(\pi/6) = 1/2$

$\tan(\pi/6) = 1/\sqrt{3} = \sqrt{3}/3$

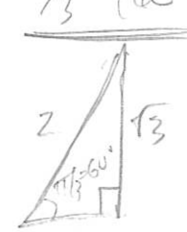
$\pi/4$ (45°)

$\cos(\pi/4) = 1/\sqrt{2} = \sqrt{2}/2$

$\sin(\pi/4) = 1/\sqrt{2} = \sqrt{2}/2$

$\tan(\pi/4) = 1$

$\pi/3$ (60°)



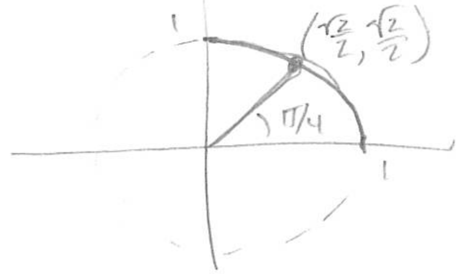
$\cos(\pi/3) = 1/2$

$\sin(\pi/3) = \sqrt{3}/2$

$\tan(\pi/3) = \sqrt{3}$

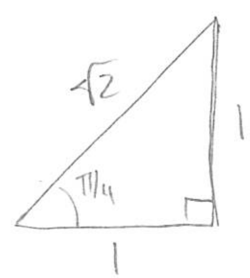
- We can use the three prototype triangles as an alternative for navigating the first quadrant of the unit circle.

Ex: <u>Unit circle</u>



$$\cos(\pi/4) = x\text{-coordinate} = \boxed{\frac{\sqrt{2}}{2}}$$
$$\sin(\pi/4) = y\text{-coordinate} = \boxed{\frac{\sqrt{2}}{2}}$$
$$\tan(\pi/4) = \frac{\sin(\pi/4)}{\cos(\pi/4)} = \frac{\sqrt{2}/2}{\sqrt{2}/2} = \boxed{1}$$

<u>Prototype Triangle</u>



$$\cos(\pi/4) = \frac{Adj}{Hyp} = \frac{1}{\sqrt{2}} = \boxed{\frac{\sqrt{2}}{2}}$$
$$\sin(\pi/4) = \frac{Opp}{Hyp} = \frac{1}{\sqrt{2}} = \boxed{\frac{\sqrt{2}}{2}}$$
$$\tan(\pi/4) = \frac{Opp}{Adj} = \frac{1}{1} = \boxed{1}$$

$\boxed{\text{WS}}$ #3,4,5 Working with Prototype Triangles

- We note here, again, that four of the trig functions are defined in terms of $\cos(\theta)$ and $\sin(\theta)$. These are called <u>identities</u>. There are many varieties of these identities:

<u>Reciprocal Identities</u>

$$\sin(\theta) = \frac{1}{\csc(\theta)} \qquad \sec(\theta) = \frac{1}{\cos(\theta)} \qquad \tan(\theta) = \frac{1}{\cot(\theta)}$$
$$\cos(\theta) = \frac{1}{\sin(\theta)} \qquad \csc(\theta) = \frac{1}{\sin(\theta)} \qquad \cot(\theta) = \frac{1}{\tan(\theta)}$$

Quotient Identities: $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$ $\cot(\theta) = \frac{\cos(\theta)}{\sin(\theta)}$

- we note again that on the unit circle, defined by $x^2+y^2=1$, the x-coordinate is $\cos(\theta)$ and the y coordinate is $\sin(\theta)$. Hence, for all θ, we must have:

$$x^2+y^2=1 \qquad \begin{array}{l} x=\cos(\theta) \\ y=\sin(\theta) \end{array}$$

$$\Rightarrow (\cos(\theta))^2+(\sin(\theta))^2=1$$

$$\Rightarrow \boxed{\cos^2(\theta)+\sin^2(\theta)=1}$$ Pythagorean's Identity.

- Note: the notation $\cos^2(\theta)$ means $(\cos(\theta))^2$ or $\cos(\theta)\cos(\theta)$.
- From this identity, we can create two more identities:

$$\sin^2(\theta)+\cos^2(\theta)=1$$

$$\Rightarrow \frac{\sin^2(\theta)}{\sin^2(\theta)}+\frac{\cos^2(\theta)}{\sin^2(\theta)}=\frac{1}{\sin^2(\theta)} \Rightarrow \boxed{1+\cot^2(\theta)=\csc^2(\theta)}$$

$$\Rightarrow \frac{\sin^2(\theta)}{\cos^2(\theta)}+\frac{\cos^2(\theta)}{\cos^2(\theta)}=\frac{1}{\cos^2(\theta)} \Rightarrow \boxed{\tan^2(\theta)+1=\sec^2(\theta)}$$

- We have

Pythagorean Identities

$$\sin^2(\theta)+\cos^2(\theta)=1$$
$$1+\cot^2(\theta)=\csc^2(\theta)$$
$$\tan^2(\theta)+1=\sec^2(\theta)$$

- Identities can allow us to evaluate other trig values of angles:

Ex: Suppose we know $\tan(\theta) = \frac{1}{3}$. Then we can easily find $\cot(\theta)$ and $\sec(\theta)$ without knowing the angle θ.

$$\tan\theta = \frac{1}{3} \Rightarrow \cot(\theta) = \frac{1}{\tan(\theta)} = \frac{1}{1/3} = \boxed{3}$$

$$\Rightarrow \sec^2(\theta) = 1 + \tan^2(\theta)$$

$$\Rightarrow \sec^2(\theta) = 1 + \left(\frac{1}{3}\right)^2$$

$$\Rightarrow \sec^2(\theta) = 1 + \frac{1}{9}$$

$$\Rightarrow \sec^2(\theta) = \frac{10}{9}$$

$$\Rightarrow \boxed{\sec(\theta) = \frac{\sqrt{10}}{9}}$$

- We can also use identities to create other identities:

Ex: Prove that $(1+\sin\theta)(1-\sin(\theta)) = \cos^2(\theta)$

LHS: $(1+\sin\theta)(1-\sin(\theta)) = 1 - \cancel{\sin(\theta)} + \cancel{\sin(\theta)} - \sin^2(\theta)$

$$= 1 - \sin^2(\theta)$$

$$= \sin^2(\theta) + \cos^2(\theta) - \sin^2(\theta)$$

$$= \cos^2(\theta) \checkmark \quad \text{RHS}$$

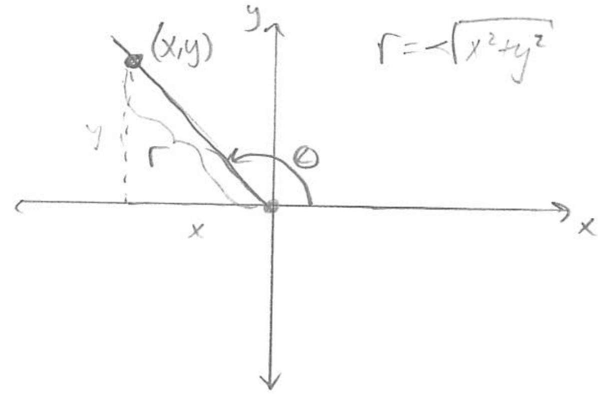
$\boxed{\text{WS}}$ #6 Working with identities

Section 1.4: Trig Functions of Any Angle

- We've study the trig functions applied to any acute angle θ. Now, we can apply the trig functions to any angle drawn on the Cartesian Plane.

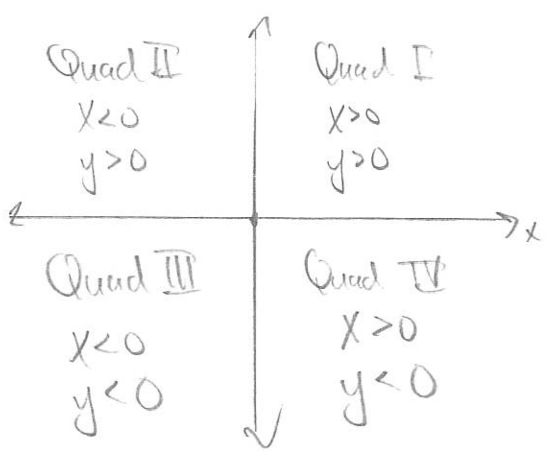
- Let θ be any angle corresponding to the point (x,y), where (x,y) lies on the terminal side of θ. Then we define the six trig functions as follows

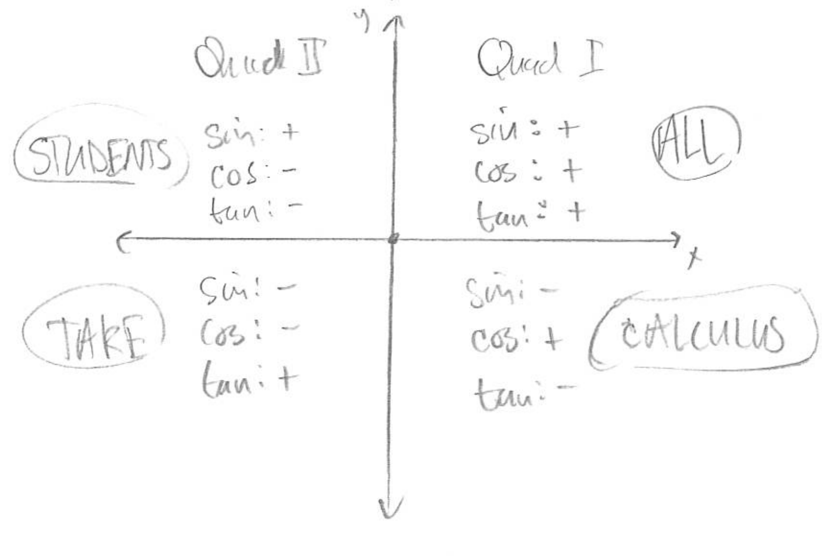
$$\cos(\theta) = \frac{x}{r} \qquad \sec(\theta) = \frac{r}{x}$$
$$\sin(\theta) = \frac{y}{r} \qquad \csc(\theta) = \frac{r}{y}$$
$$\tan(\theta) = \frac{y}{x} \qquad \cot(\theta) = \frac{x}{y}$$



$\boxed{\text{WS}}$ #1, a,b Working with signed trig functions.

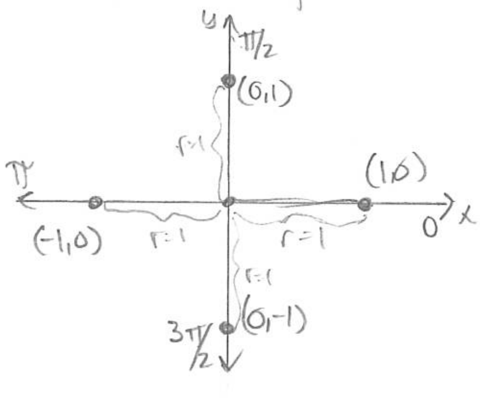
- Using the definitions of the trig functions for any angle, we see that the signs of the trig functions switch depending on the quadrant.





$\boxed{WS}$ #2 Working with quadrants and trig functions.

- Using this definition of the trig functions, we can evaluate the trig functions at the four axes angles:



$\cos(0) = \frac{x}{r} = \frac{1}{1} = 1$ $\qquad \sec(0) = 1$

$\sin(0) = \frac{y}{r} = \frac{0}{1} = 0$ $\qquad \csc(0) = \frac{1}{0} = \text{DNE}$

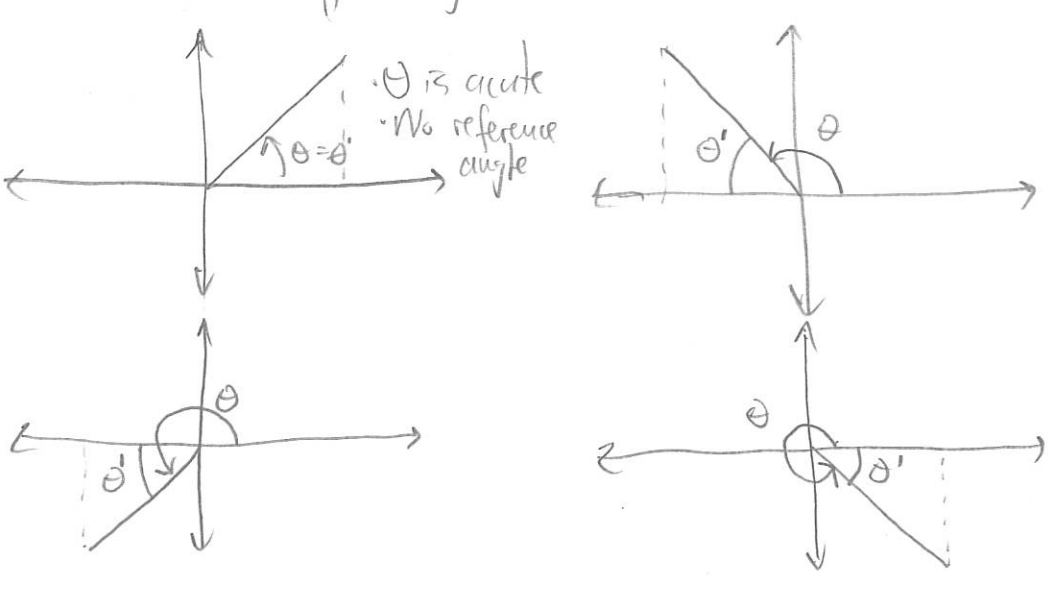
$\tan(0) = \frac{y}{x} = \frac{0}{1} = 0$ $\qquad \cot(0) = \frac{1}{0} = \text{DNE}$

- There are values of θ for which some of the trig functions are undefined.

$\boxed{WS}$ #3 Find the trig functions of $\theta = \frac{3\pi}{2}$.

- To help us evaluate trig functions at other angles, we define what is called a <u>reference angle</u>.

  - Let θ be any angle, then the <u>reference angle</u> is the acute angle formed by the terminal side of θ and the x-axis. We denote this angle by θ'.



- To find the trigonometric value of any angle θ:
  1.) Determine the function value of the reference angle θ'.
  2.) Depending on the quadrant in which θ lies, affix the appropriate sign to the function value.

$\boxed{WS}$ #4, #5 working with Reference Angles

- We note that we can only find the trig value exactly if the angle corresponds to a reference angle of $0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}$. In all other cases, we have to use the calculator.

$\boxed{WS}$ #6 Finding trig functions with calculators.

Section 1.5: Graphs of Sine & Cosine.

- We can graph the functions $\sin(\theta)$ and $\cos(\theta)$ by defining the y variable as the function output and the input angle θ as x.
- That is, consider all points (x,y) that satisfy the equations

$$y = \cos(x) \qquad \text{and} \qquad y = \sin(x).$$

- To graph these functions, we consider the five key points when

$$x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi.$$

$\boxed{WS}$ #1 a,b Graph sine and cosine.

- Notes about the sine and cosine curve.
  - The sine curve is symmetric about the origin (rotational symmetry)

- The cosine graph is symmetric about the y-axis.
- Since $\sin(x)$ and $\cos(x)$ repeat after 2π, we say that the <u>period</u> of $\sin(x)$ and $\cos(x)$ is 2π.
- Domain of $\sin(x)$ & $\cos(x)$ is all real numbers, Range is $[-1,1]$.
- Key Points of $\sin(x)$:

$(0,0)$	$(\frac{\pi}{2},1)$	$(\pi,0)$	$(\frac{3\pi}{2},-1)$	$(2\pi,0)$
x-int	max	x-int	min	x-int

- Key Points of $\cos(x)$:

$(0,1)$	$(\frac{\pi}{2},0)$	$(\pi,-1)$	$(\frac{3\pi}{2},0)$	$(2\pi,1)$
max	x-int	min	x-int	max

- The <u>amplitude</u> of the sine and cosine graphs is determined by the coefficient in front of the function.
  - $y=\sin(x) \longrightarrow y=1\sin(x)$ max and min at ±1.
  - $y=\cos(x) \longrightarrow y=1\cos(x)$ max and min at ±1

The functions $y=a\sin(x)$ and $y=a\cos(x)$ have an <u>amplitude</u> of $|a|$ which represents the max/min value that the function attains.

$\boxed{\text{Amplitude} = |a|}$ only effects y-coords.

$\boxed{\text{WS}}$ #2 Compare amplitudes.

- The amplitude simply refers to the vertical stretch transformation of the graph.

- The <u>period</u> of the sine and cosine graphs represents the span on the x-axis for which the graph starts to repeat itself.

  - $y=\sin(x) \longrightarrow y=\sin(1x) \longrightarrow$ period is $\frac{2\pi}{1}=2\pi$
  - $y=\cos(x) \longrightarrow y=\cos(1x) \longrightarrow$ period is $\frac{2\pi}{1}=2\pi$

- The functions $y=\sin(bx)$ and $y=\cos(bx)$ have a <u>period</u> of

$$\boxed{\text{Period}=\frac{2\pi}{b}}$$

only effects x-coords

WS #1 Working with different periods.

- The last thing we can consider is a <u>phase shift</u>, which is essentially a horizontal shift along the x-axis.

$$y=\sin(x-c) \quad \text{or} \quad y=\cos(x-c)$$

- If c is positive, shift right c units.
- If c is negative shift left c units.

$\Big\}$ Only effects x-coords

WS #2 Working with basic shifts.

- The most general form of a trig function is the following:

$$y=a\sin(bx-c) \qquad y=a\cos(bx-c)$$

These functions have the following characteristics:

- Amplitude $= |a|$
- Period $= \frac{2\pi}{b}$
- Fundamental Cycle $\begin{cases} \text{Starts at } x = \frac{c}{b} \\ \text{Ends at } x = \frac{2\pi + c}{b} \end{cases}$

Note: for graphing these functions, always find the starting and ending part of the Fundamental Cycle.

$\boxed{\text{WS}}$ #1 – #2, #1. Graphing trig functions

- Finally, we could add vertical shifts to these graphs

$$y = a\sin(bx - c) + d \qquad y = a\cos(bx - c) + d$$

effects only y-coords

The addition of d simply moves the graph up or down.

- In general, the first and last of the 5 key points get translated to the following:

$\underline{y = \sin(x)}$

$$\left.\begin{aligned} (0,0) &\longrightarrow \left(\tfrac{c}{b}, d\right) \\ (2\pi, 0) &\longrightarrow \left(\tfrac{2\pi + c}{b}, d\right) \end{aligned}\right\} y = a\sin(bx - c) + d$$

$\underline{y = \cos(x)}$:

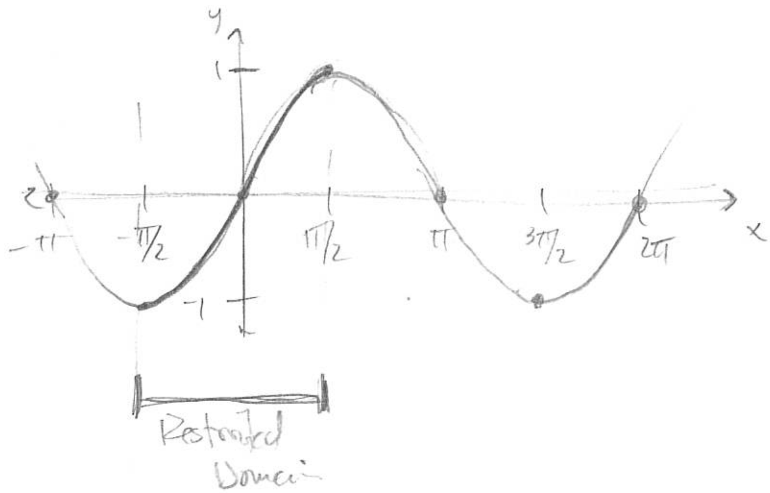
$$(0,1) \longrightarrow \left(\tfrac{c}{b}, a + d\right)$$

$$(2\pi, 1) \longrightarrow \left(\tfrac{2\pi + c}{b}, a + d\right)$$

Section 1.7 - The Inverse Trig Functions

- Now that we know a little bit more about function inverses, we can define the inverse trig functions.

- We can start with the sine function, $f(x)=\sin(x)$. Obviously, $f(x)=\sin(x)$ does not satisfy the horizontal line test. Therefore, in order to define an inverse function, we have to restrict the domain:

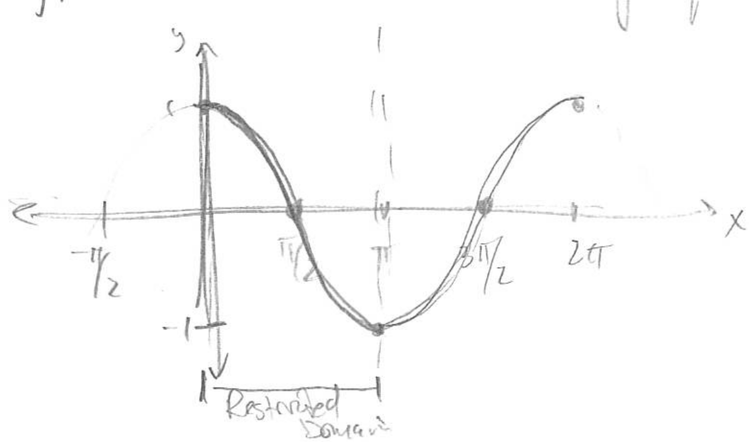


- Sine fails HLT on $(-\infty,\infty)$
- Sine passes HLT on $[-\pi/2, \pi/2]$

we define the inverse of sine on $[-\pi/2, \pi/2]$:

		Domain:	Range
Original:	$f(x)=\sin(x)$	$[-\pi/2, \pi/2]$	$[-1,1]$
Inverse:	$f^{-1}(x)=\arcsin(x)$ $\left(f^{-1}(x)=\sin^{-1}(x)\right)$	$[-1,1]$	$[-\pi/2, \pi/2]$

- Similarly, for cosine, consider the graph of $f(x)=\cos(x)$.



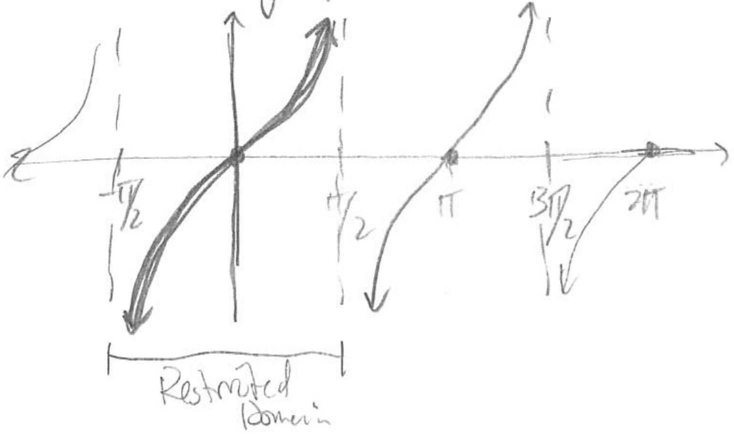
- Cosine fails HLT on $(-\infty,\infty)$
- Cosine passes HLT on $[0,\pi]$

- We define the inverse of cosine on $[0,\pi]$:

		Domain	Range
Original:	$f(x)=\cos(x)$	$[0,\pi]$	$[-1,1]$
Inverse:	$f^{-1}(x)=\arccos(x)$	$[-1,1]$	$[0,\pi]$

$(f^{-1}(x)=\cos^{-1}(x))$.

- Finally, consider the graph of $f(x)=\tan(x)$



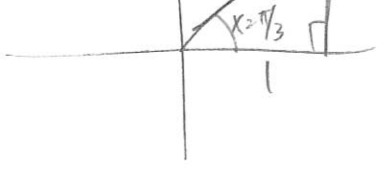
- tangent fails HLT on $(-\infty,\infty)$
- tangent passes HLT on $[-\pi/2,\pi/2]$

- we define the inverse tangent function on $[-\pi/2,\pi/2]$:

		Domain	Range
Original:	$f(x)=\tan(x)$	$[-\pi/2,\pi/2]$	$(-\infty,\infty)$
Inverse:	$f^{-1}(x)=\arctan(x)$	$(-\infty,\infty)$	$[-\pi/2,\pi/2]$.

- Now, how do we evaluate these functions? Consider evaluating the sine function:

Ex: Suppose $x=\pi/3$, an angle in the first quadrant, then we can evaluate the sine of this angle to obtain a ratio:



$$\sin(\pi/3)=\frac{\sqrt{3}}{2}$$

Input: any angle Output: ratio inside $[-1,1]$

To evaluate the arcsine function, we reverse this process:

$\arcsin(x)$ Input: ratio inside $[-1,1]$
Output: angle inside $[-\pi/2, \pi/2]$

Therefore, $\arcsin\left(\frac{\sqrt{3}}{2}\right)$ ⟵ "what angle such that when I apply sine to it yields $\frac{\sqrt{3}}{2}$?"

$$\arcsin\left(\frac{\sqrt{3}}{2}\right) = \boxed{\frac{\pi}{3}}$$

↳ Since $\frac{\pi}{3}$ is inside the range $[-\pi/2, \pi/2]$, this is the correct answer.

$\boxed{WS}$ #2, #3, #4 Working with evaluating inverse trig functions.
HW

· By the property of inverse functions, if we compose a trig function with its inverse, they should cancel out, but we have to be very careful.

- $\underbrace{\arcsin(\sin(x)) = x}_{\text{only if } x \text{ is in } [-\pi/2, \pi/2]}$ and $\underbrace{\sin(\arcsin(x)) = x}_{\text{only if } x \text{ is in } [-1,1]}$

- $\underbrace{\arccos(\cos(x)) = x}_{\text{only if } x \in [0,\pi]}$ and $\underbrace{\cos(\arccos(x)) = x}_{\text{only if } x \text{ is in } [-1,1]}$

- $\underbrace{\arctan(\tan(x)) = x}_{\text{only if } x \text{ is in } [-\pi/2, \pi/2]}$ and $\underbrace{\tan(\arctan(x)) = x}_{\text{only if } x \in (-\infty,\infty)}$

Ex: $\arcsin(\sin(\frac{\pi}{3}))$

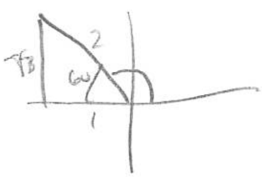
Since $\pi/3 \in [-\pi/2, \pi/2]$, we have

$$\arcsin(\sin(\tfrac{\pi}{3})) = \frac{\pi}{3}.$$

Ex: $\arcsin(\sin(\frac{2\pi}{3}))$

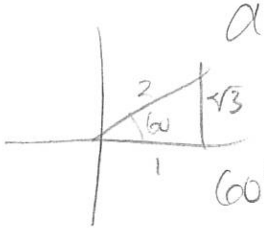
Since $2\pi/3$ is outside $[-\pi/2, \pi/2]$, we must first evaluate $\sin(\frac{2\pi}{3})$

$$\sin(2\pi/3) = \frac{\sqrt{3}}{2}$$



Now, we do $\arcsin(\frac{\sqrt{3}}{2})$. What angle in the first or 4th quadrant gives $\frac{\sqrt{3}}{2}$ when sine is applied?

$$\arcsin(\tfrac{\sqrt{3}}{2}) = \boxed{\frac{\pi}{3}}$$



$60° = \pi/3$

$\boxed{\text{WS}}$ #5 a–e. working with function compositions