MAT 362 - Operations Research II

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CHAPTER 14 - Game Theory

Section 14.1: Two-Person Zero-Sum and Constant-Sum Games: Saddle Points

- In every problem we have encountered thus far in this class, we have been making decisions as a single decision-maker — we've not taken into consideration the action of other outside decision-makers who may or may not have a similar objective.
- Game theory deals with decision situations in which two or more intelligent opponents with conflicting objectives are attempting to outdo one another.
- We'll consider two-player zero-sum games:
  - Two players (opponents) are each able to identify a strategy from a list of known strategies.
  - Each strategy has an associated payoff that one player will pay to the other. A gain by one player is an identical loss to the other player — zero-sum game.
  - Player A (the row player) has a total of m strategies to choose from: $A_1, A_2, \ldots, A_m$. Player B (the column player) has n strategies $B_1, B_2 \ldots B_n$.

- The game is constructed via a reward matrix in terms of the payoff to player A (the row player):

Player B strategies:

Player A strategies

$$\begin{array}{c|cccc} & B_1 & B_2 & \cdots & B_n \\ \hline A_1 & P_{11} & P_{12} & \cdots & P_{1n} \\ A_2 & P_{21} & P_{22} & \cdots & P_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_m & P_{m1} & P_{m2} & \cdots & P_{mn} \end{array}$$

P_{ij} = payoff to player A from player A using strategy A_i and player B using strategy B_j

- To analyze the best strategy for the game, we make a key assumption:

  Each player chooses a best strategy, given that the opponent knows the strategy they have chosen.

Under this assumption, each player will typically choose the "best of the worst" strategies.

- Player A, who is trying to max the payout to A, will choose the maximum value from the row minimums

$$\max_i \underbrace{\left(\min_j \left(P_{ij} \right) \right)}_{\substack{\text{set of all row minimums i.e.} \\ \text{set of all minimum gains for Player A}}} \longleftarrow \text{Maximin}$$

- Player B, who is trying to min the payout to A, will chose the minimum value from the column maximums

$$\min_j \left(\max_i \left(P_{ij} \right) \right) \leftarrow \text{Minimax}$$

set of all column maximums i.e.

set of all maximum losses for Player B

WS 14.1 #1-#4 work with a zero sum and constant sum games

- If the values of the minimax and maximin are equal, then the value of the game is this common value, and both players are using a pure saddle point solution.

Section 14.2: Two Person Zero-Sum Games: Randomized Strategies, Domination, and Graphical Solution

- In the case of a pure saddle point solution, both players choose a single strategy that yields the same value (maximin = minimax). This is called a pure strategy (i.e. Player A plays A_2 with probability 1 and Player B plays B_3 with probability 1) as opposed to a mixed strategy, where a player can play a variety of strategies with different probabilities.
- The optimal saddle point solution means that if either player deviates from the strategy, the other player will have an advantage or a worse payout will ensue.

- A saddle point solution need not be pure.

$\boxed{WS}$ 14.2 #1 work with the heads/tails game.

- In any game, the value of the game, denoted by v, is between the maximin and minimax:

$$\max_i\left(\min_j(p_{ij})\right) \leq V \leq \min_j\left(\max_i(p_{ij})\right)$$

If a pure saddle point solution doesn't exist, we employ a mixed strategy:

<u>Player A</u> : x_i = probability that strategy A_i is played

<u>Player B</u> : y_j = probability that strategy B_j is played.

- If at least one player has exactly 2 strategies, then the problem can be solved graphically.
- Note, in a mixed strategy game, we assume the opponent will still choose a pure strategy!

$\boxed{WS}$ 14.2 #1, #2 work with mixed strategy games.

- Essentially, we plot A's expected payoff based on B's pure strategies.

Section 14.3: Linear Programming and Zero Sum Games

- Using the idea of a mixed strategy, we can formulate an LP for each player in a general two-player zero-sum game:

$$
\begin{array}{c|c}
 & \text{Player B strategies:} \\
 & \begin{array}{cccc} B_1 & B_2 & \cdots & B_n \end{array} \\
\text{Player A strategies} \begin{array}{c} A_1 \\ A_2 \\ \vdots \\ A_m \end{array} & \boxed{\begin{array}{cccc} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ p_{m1} & p_{m2} & \cdots & p_{mn} \end{array}}
\end{array}
$$

Player A's Goal

B's Strategy	A's Expectation
B_1:	$\sum_{i=1}^{m} p_{i1} x_i$
B_2:	$\sum_{i=1}^{m} p_{i2} x_i$
$\vdots$	$\vdots$
B_n:	$\sum_{i=1}^{m} p_{in} x_i$

Compute maximin

$$\max_{x_i} \Big\{ \underbrace{\min\left(\textstyle\sum_i p_{i1} x_i, \ldots, \sum_i p_{in} x_i\right)}_{\text{Define this as } v} \Big\}$$

Player A's Obj. Fun

Maximize $z = v$

Player B's Goal

B's Strategy	A's Expectation
A_1:	$\sum_{j=1}^{n} p_{1j} y_j$
A_2:	$\sum_{j=1}^{n} p_{2j} y_j$
$\vdots$	$\vdots$
A_m:	$\sum_{j=1}^{n} p_{mj} y_j$

Compute minimax

$$\min_{y_j} \Big\{ \underbrace{\max\left(\textstyle\sum_j p_{1j} y_j, \ldots, \sum_j p_{mj} y_j\right)}_{\text{Define this as } w} \Big\}$$

Player B's Obj. Fun

Minimize $z = w$

Player A's Constraints

Since $v = \min\left(\sum_i p_{i1} x_i, \ldots, \sum_i p_{in} x_i\right)$

we have

$$v \le \sum_{i=1}^{m} p_{i1} x_i$$

$$v \le \sum_{i=1}^{m} p_{i2} x_i$$

$$\vdots$$

$$v \le \sum_{i=1}^{m} p_{in} x_i$$

$$\sum_{i=1}^{m} x_i = 1$$

$$x_i \ge 0, \quad v = \text{urs}$$

The optimal value v is Player A's value of the game.

Player B's Constraints

Since $w = \max\left(\sum_j p_{1j} y_j, \ldots, \sum_j p_{mj} y_j\right)$

we have

$$\sum_{j=1}^{n}$$

$$\sum_{j=1}^{n}$$

$$\vdots$$

$$\sum_{j=1}^{n}$$

$$\sum_{j=1}^{n} y_j = 1$$

$$y_j \ge 0, \quad w = \text{urs}$$

The optimal value w is Player B's value of the game.

At optimality, $v = w$

Player A's LP is the <u>dual problem</u> to Player B's LP!

$\boxed{\text{WS}}$ 14.3 #1–#3 working with formulating a Zero-Sum Game.