§9.5: The Comparison, Ratio, and Root Tests

1.] Determine if the following series converge or diverge using the Comparison Test:

a.) $\sum_{k=1}^{\infty} \frac{1}{k^2+4}$

Let $a_k = \frac{1}{k^2+4}$ and $b_k = \frac{1}{k^2}$

Then $k^2+4 \geq k^2$

$\Rightarrow \frac{1}{k^2+4} \leq \frac{1}{k^2}$

$\Rightarrow a_k \leq b_k$.

Since $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{1}{k^2}$ converges $(p=2>1)$ we know $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{1}{k^2+4}$ also $\boxed{\text{Converges}}$ by comparison.

b.) $\sum_{k=1}^{\infty} \frac{k^3}{2k^4-1}$

$\frac{k^3}{2k^4-1} \geq \frac{k^3}{2k^4}$

Hence, $\sum_{k=1}^{\infty} \frac{k^3}{2k^4-1} \geq \sum_{k=1}^{\infty} \frac{1}{2k} = \frac{1}{2}\sum_{k=1}^{\infty} \frac{1}{k}$

$\Rightarrow \frac{k^3}{2k^4-1} \geq \frac{1}{2k}$ for all k. Since $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges (Harmonic), $\sum_{k=1}^{\infty} \frac{k^3}{2k^4-1}$ also $\boxed{\text{diverges}}$

c.) $\sum_{k=2}^{\infty} \frac{\ln(k)}{k^2}$

Note, $\ln(k) \leq \sqrt{k}$

so that $\frac{\ln(k)}{k^2} \leq \frac{\sqrt{k}}{k^2}$

$\Rightarrow \frac{\ln(k)}{k^2} \leq \frac{1}{k^{3/2}}$

Hence, $\sum_{k=1}^{\infty} \frac{\ln(k)}{k^2} \leq \sum_{k=1}^{\infty} \frac{1}{k^{3/2}}$ and since $\sum_{k=1}^{\infty} \frac{1}{k^{3/2}}$ converges $(p=3/2>1)$, $\sum_{k=1}^{\infty} \frac{\ln(k)}{k^2}$ also $\boxed{\text{converges}}$ by comparison.

2.] Determine if the following series converge or diverge using the Limit Comparison Test:

a.) $\sum_{k=2}^{\infty} \frac{1}{k^2-1}$

Let $a_k = \frac{1}{k^2-1}$ and $b_k = \frac{1}{k^2}$, then

$$\lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{\frac{1}{k^2-1}}{\frac{1}{k^2}} = \lim_{k\to\infty} \frac{k^2}{k^2-1} = \lim_{k\to\infty} \frac{1}{1-\frac{1}{k^2}} = \frac{1}{1-0} = 1$$

Since $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges $(p=2>1)$, $\sum_{k=1}^{\infty} \frac{1}{k^2-1}$ also $\boxed{\text{converges}}$

b.) $\sum_{k=1}^{\infty} \frac{k^4-2k^2+3}{2k^6-k+5}$

Let $a_k = \frac{k^4-2k^2+3}{2k^6-k+5}$ and $b_k = \frac{1}{k^2}$, then

$$\lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{k^4-2k^2+3}{2k^6-k+5} \cdot \frac{k^2}{1} = \lim_{k\to\infty} \frac{k^6-2k^4+3k^2}{2k^6-k+5} = \frac{1}{2}$$

Since $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges, $\sum_{k=1}^{\infty} \frac{k^4-2k^2+3}{2k^6-k+5}$ also $\boxed{\text{converges.}}$

c.) $\sum_{k=3}^{\infty} \frac{\ln(k)}{k^2}$

Let $a_k = \frac{\ln(k)}{k^2}$ and $b_k = \frac{1}{k^{3/2}}$, then

$$\lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{\ln(k)}{k^2} \cdot \frac{k^{3/2}}{1} = \lim_{k\to\infty} \frac{\ln(k)}{k^{1/2}} = \lim_{k\to\infty} \frac{\frac{1}{k}}{\frac{1}{2k^{1/2}}} = \lim_{k\to\infty} \frac{1}{2k^{1/2}} = 0$$

Since $\sum_{k=1}^{\infty} \frac{1}{k^{3/2}}$ converges $(p=\frac{3}{2}>1)$, $\sum_{k=1}^{\infty} \frac{\ln(k)}{k^2}$ also $\boxed{\text{converges.}}$

3.] Determine if the following series converge or diverge using the Ratio Test:

$a.)\ \sum_{k=1}^{\infty} \frac{8^k}{k!}$

$\lim \frac{a_{k+1}}{a_k} = \lim_{k\to\infty} \frac{8^{k+1}}{(k+1)!} \cdot \frac{k!}{8^k} = \lim_{k\to\infty} \frac{8}{k+1} = 0$

$a_{k+1} = \frac{8^{k+1}}{(k+1)!}$

By Ratio Test, $\sum_{k=1}^{\infty} \frac{8^k}{k!}$ $\boxed{\text{converges}}$.

$b.)\ \sum_{k=1}^{\infty} \frac{k^k}{k!}$

$a_{k+1} = \frac{(k+1)^{k+1}}{(k+1)!}$

$\lim_{k\to\infty} \frac{(k+1)^{k+1}}{(k+1)!} \cdot \frac{k!}{k^k} = \lim_{k\to\infty} \frac{(k+1)^k(k+1)\,k!}{(k+1)!\,k^k} = \lim_{k\to\infty} \frac{(k+1)^k}{k^k} = \lim_{k\to\infty} \left(\frac{k+1}{k}\right)^k = \lim_{k\to\infty}\left(1+\frac{1}{k}\right)^k = e > 1$

$\boxed{\text{Diverges by Ratio Test}}$

4.] Determine if the following series converge or diverge using the Root Test:

$a.)\ \sum_{k=1}^{\infty} \left(\frac{3k^2-1}{8k^2+k}\right)^k$

$\lim_{k\to\infty} \sqrt[k]{a_k} = \lim_{k\to\infty} \left(\frac{3k^2-1}{8k^2+k}\right) = \frac{3}{8} < 1$

$\sqrt[k]{a_k} = \left(\frac{3k^2-1}{8k^2+k}\right)$

$\boxed{\text{Converges by Root Test}}$

$b.)\ \sum_{k=1}^{\infty} \frac{2^k}{k^9}$

$\lim_{k\to\infty} \sqrt[k]{a_k} = \lim_{k\to\infty} \frac{2}{k^{9/k}} = 2\lim_{k\to\infty} \left(\frac{1}{k^{1/k}}\right)^9 = 2 > 1$

$\sqrt[k]{a_k} = \frac{2}{k^{9/k}}$

$\boxed{\text{Diverges by Root Test}}$

5.] Determine if the following series converge or diverge:

$a.)\ \sum_{k=1}^{\infty} \left(1+\frac{2}{k}\right)^k$

Div Test: $\lim_{k\to\infty} \left(1+\frac{2}{k}\right)^k = e^2 \neq 0$

$\boxed{\text{Diverges by Divergence Test}}$

$b.)\ \sum_{k=1}^{\infty} \frac{2^k k!}{k^k}$

$a_{k+1} = \frac{2^{k+1}(k+1)!}{(k+1)^{k+1}}$

$\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \lim_{k\to\infty} \frac{2^{k+1}(k+1)!}{(k+1)^{k+1}} \frac{k^k}{2^k\,k!}$

$= \lim_{k\to\infty} \frac{2\cdot(k+1)!\,k^k}{(k+1)^k(k+1)\,k!}$

$= \lim_{k\to\infty} \frac{2\,k^k}{(k+1)^k}$

$= \lim_{k\to\infty} 2\left(\frac{k}{k+1}\right)^k = \lim_{k\to\infty} \frac{2}{\left(\frac{k+1}{k}\right)^k} = \lim_{k\to\infty} \frac{2}{\left(1+\frac{1}{k}\right)^k} = \frac{2}{e} < 1$

$\boxed{\text{Converges}}$