§9.4: The Divergence & Integral Tests

1.] Determine if the following series are divergent or state that the Divergence Test is inconclusive.

a.) $\sum_{k=0}^{\infty} \frac{k}{k+1}$ $\lim_{k\to\infty} \frac{k}{k+1} = \lim_{k\to\infty} \frac{1}{1+1/k} = 1 \neq 0$ $\boxed{\text{Diverges}}$ by Divergence Test

b.) $\sum_{k=1}^{\infty} \frac{1+3^k}{2^k}$ $\lim_{k\to\infty} \frac{1+3^k}{2^k} = \lim_{k\to\infty} \left(\frac{1}{2}\right)^k + \left(\frac{3}{2}\right)^k = 0 + \infty \neq 0$ $\boxed{\text{Diverges}}$

c.) $\sum_{k=1}^{\infty} \frac{1}{k}$ $\lim_{k\to\infty} \frac{1}{k} = 0$ $\boxed{\text{Inconclusive}}$

d.) $\sum_{k=1}^{\infty} \frac{1}{k^2}$ $\lim_{k\to\infty} \frac{1}{k^2} = 0$ $\boxed{\text{Inconclusive}}$

2.] Determine if the following series converge or diverge using the Integral Test:

a.) $\sum_{k=1}^{\infty} \frac{k}{k^2+1}$ $\int_1^{\infty} \frac{x}{x^2+1}\,dx = \lim_{b\to\infty} \ln|x^2+1| \Big|_1^b = \lim_{b\to\infty} \ln(b^2+1) - \ln|2| = \ln(\infty) = \boxed{\infty}$

$u = x^2+1$
$du = 2x\,dx$ $\int \frac{x}{x^2+1}\,dx = \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln(u) = \frac{1}{2}\ln|x^2+1|$ $\boxed{\text{Diverges.}}$

b.) $\sum_{k=3}^{\infty} \frac{1}{\sqrt{2k-5}}$ $\int_3^{\infty} \frac{1}{\sqrt{2x-5}}\,dx = \lim_{b\to\infty} \sqrt{2x-5}\Big|_3^b = \lim_{b\to\infty} \sqrt{2b-5} - 1 = \boxed{\infty}$ $\boxed{\text{Diverges}}$

$u = 2x-5$
$du = 2dx$ $\int \frac{1}{\sqrt{2x-5}}\,dx = \frac{1}{2}\int \frac{1}{\sqrt{u}}\,du = \frac{1}{2}\int u^{-1/2}\,du = u^{1/2} = \sqrt{2x-5}$

c.) $\sum_{k=0}^{\infty} \frac{1}{k^2+4}$ $\int_0^{\infty} \frac{1}{x^2+4}\,dx = \lim_{b\to\infty} \frac{1}{2}\arctan\left(\frac{x}{2}\right)\Big|_0^b = \lim_{b\to\infty} \frac{1}{2}\arctan\left(\frac{b}{2}\right) - 0$
$= \frac{1}{2}\left(\frac{\pi}{2}\right) = \boxed{\frac{\pi}{4}}$ $\boxed{\text{Converges}}$

$\int \frac{1}{x^2+4}\,dx = \frac{1}{4}\int \frac{1}{1+\frac{x^2}{4}}\,dx = \frac{1}{4}\int \frac{1}{1+\left(\frac{x}{2}\right)^2}\,dx = \frac{1}{4}\,2\arctan\left(\frac{x}{2}\right) = \frac{1}{2}\arctan\left(\frac{x}{2}\right)$

3.] Determine if the following series converge or diverge using the p-Test:

$a.)\ \sum_{k=1}^{\infty} \frac{1}{k^{10}}$ $p=10>1 \to \boxed{\text{Converges}}$

$b.)\ \sum_{k=1}^{\infty} \frac{1}{k^{-1/5}}$ $p=-\frac{1}{5}<1 \to \boxed{\text{Diverges}}$

$c.)\ \sum_{k=1}^{\infty} \frac{1}{\sqrt{k^3}} = \sum_{k=1}^{\infty} \frac{1}{k^{3/2}}$ $p=\frac{3}{2}>1 \to \boxed{\text{Converges}}$

4.] Determine if the following series converge or diverge:

$a.)\ \sum_{k=1}^{\infty} \frac{\sqrt{k^2+1}}{k}$

Div Test: $\lim_{k\to\infty} \frac{\sqrt{k^2+1}}{k} = \lim_{k\to\infty} \sqrt{\frac{k^2+1}{k^2}} = \lim_{k\to\infty} \sqrt{1+\frac{1}{k^2}}$

$= \sqrt{1+0}$

$= 1 \neq 0$

$\boxed{\text{Diverges}}$

$b.)\ \sum_{k=1}^{\infty} \frac{k}{e^k}$ Div Test: $\lim_{k\to\infty} \frac{k}{e^k} = \lim_{k\to\infty} \frac{1}{e^k} = 0$ $\to$ Inconclusive.

$$\int_1^{\infty} \frac{x}{e^x}\,dx = \int_1^{\infty} xe^{-x}\,dx = -xe^{-x}\Big|_1^{\infty} + \int_1^{\infty} e^{-x}\,dx = -xe^{-x} - e^{-x}\Big|_1^{\infty} = \lim_{b\to\infty}(-be^{-b}-e^{-b}) - (-e^{-1}-e^{-1})$$

$= 0 + 2e^{-1} = \frac{2}{e}$ $\boxed{\text{Converges}}$

$c.)\ \sum_{k=1}^{\infty} k^{1/k}$

Div Test: $\lim_{k\to\infty} k^{1/k} = \lim_{k\to\infty} e^{\ln(k^{1/k})}$

$= \lim_{k\to\infty} e^{\frac{1}{k}\ln(k)}$

$= \lim_{k\to\infty} e^{\frac{\ln(k)}{k}}$

$= \lim_{k\to\infty} e^{\frac{1/k}{1}}$ (L'Hopitals)

$= \lim_{k\to\infty} e^{\frac{1}{k}}$

$= e^0$

$= 1 \neq 0$

$\boxed{\text{Diverges}}$