§9.6: Alternating Series Test & Absolute Convergence

1.] Use the Alternating Series Test to determine if the following series converge or diverge.

a.) $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2} = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k^2}$ $a_k = \frac{1}{k^2}$ Converges by AST

1.) $0 < a_{k+1} \leq a_k \Rightarrow 0 < \frac{1}{(k+1)^2} \leq \frac{1}{k^2}$ ✓ 2.) $\lim_{k\to\infty} \frac{1}{k^2} = 0$ ✓

b.) $2 - \frac{3}{2} + \frac{4}{3} - \frac{5}{4} + \cdots = \displaystyle\sum_{k=1}^{\infty} (-1)^{k+1} \frac{k+1}{k}$ $a_k = \frac{k+1}{k}$ Diverges

2.) $\lim_{k\to\infty} a_k = \lim_{k\to\infty} \frac{k+1}{k} = 1 \neq 0$

Doesn't satisfy AST

Divergence Test: $\lim_{k\to\infty} (-1)^{k+1} \frac{k+1}{k}$

$= \lim_{k\to\infty} (-1)^{k+1} \neq 0$

c.) $\displaystyle\sum_{k=2}^{\infty} \frac{(-1)^k \ln(k)}{k}$ $a_k = \frac{\ln(k)}{k}$

1.) Define $f(x) = \frac{\ln(x)}{x}$, then

$f'(x) = \frac{x(\frac{1}{x}) - \ln(x)}{x^2} = \frac{1-\ln(x)}{x^2}$

$f'(x) < 0$ when $1 - \ln(x) < 0 \Rightarrow \ln(x) > 1$
$\Rightarrow x > e$

Hence, $a_{k+1} \leq a_k$ for $k \geq 3$. ✓

Converges by AST.

2.) $\lim_{k\to\infty} \frac{\ln(k)}{k} = \lim_{k\to\infty} \frac{\frac{1}{k}}{1} = 0$ ✓

2.] Suppose $n = 9$ terms of the series $-1 + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \cdots$ are summed up. First, show this series converges. Secondly, what is the maximum error committed in approximating the value of the series.

$-1 + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \cdots = \displaystyle\sum_{k=1}^{\infty} (-1)^k \frac{1}{k!}$ $a_k = \frac{1}{k!}$

1.) $a_{k+1} = \frac{1}{(k+1)!} \leq \frac{1}{k!} = a_k$ for $k \geq 1$ ✓

2.) $\lim_{k\to\infty} \frac{1}{k!} = 0$ ✓

Converges by AST

The remainder for S_9 is bounded above by a_{10}.

Hence,

$|R_9| \leq a_{10} = \frac{1}{10!}$

$\Rightarrow |R_9| \leq 2.7557 \times 10^{-7}$

3.] Determine if the following series diverge, converge conditionally, or converge absolutely.

a.) $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\sqrt{k}}$ $\rightarrow$ $\boxed{\text{Conditionally Convergent}}$

- $\sum_{k=1}^{\infty} |a_k| = \sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} = \sum_{k=1}^{\infty} \frac{1}{k^{1/2}} \rightarrow$ Diverges ($p = \frac{1}{2} < 1$)
- $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\sqrt{k}}$. 1.) $a_{k+1} = \frac{1}{\sqrt{k+1}} \leq \frac{1}{\sqrt{k}} = a_k$ ✓ 2.) $\lim_{k\to\infty} \frac{1}{\sqrt{k}} = 0$ ✓ Converges

b.) $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\sqrt{k^3}}$ $\rightarrow$ $\boxed{\text{Absolutely Convergent}}$

- $\sum_{k=1}^{\infty} |a_k| = \sum_{k=1}^{\infty} \frac{1}{\sqrt{k^3}} = \sum_{k=1}^{\infty} \frac{1}{k^{3/2}} \rightarrow$ Converges ($p = 3/2 > 1$)

c.) $\sum_{k=1}^{\infty} \left(-\frac{1}{3}\right)^k$ $\rightarrow$ Absolutely Convergent

- $\sum_{k=1}^{\infty} |a_k| = \sum_{k=1}^{\infty} \left(\frac{1}{3}\right)^k \rightarrow$ Convergent ($r = \frac{1}{3}$, $|\frac{1}{3}| < 1$)
(Geometric Series)

$$\sum_{k=1}^{\infty} \left(-\frac{1}{3}\right)^k = \sum_{j=0}^{\infty} \left(-\frac{1}{3}\right)^{j+1} = \sum_{j=0}^{\infty} \left(-\frac{1}{3}\right)\left(-\frac{1}{3}\right)^j = \frac{-1/3}{1-(-1/3)} = \frac{-1/3}{4/3} = \boxed{-\frac{1}{4}}$$

d.) $\sum_{k=2}^{\infty} \frac{\sin(k)}{k^2} = \underbrace{\frac{\sin(2)}{4}}_{(+)} + \underbrace{\frac{\sin(3)}{9}}_{(+)} + \underbrace{\frac{\sin(4)}{16}}_{(-)} + \underbrace{\frac{\sin(5)}{25}}_{(-)} + \underbrace{\frac{\sin(6)}{36}}_{(-)} + \underbrace{\frac{\sin(7)}{49}}_{(+)} + \cdots$ Note: Cannot use AST.

- $\sum_{k=2}^{\infty} |a_k| = \sum_{k=2}^{\infty} \frac{|\sin(k)|}{k^2} \leq \sum_{k=2}^{\infty} \frac{1}{k^2}$

Since $\sum \frac{1}{k^2}$ converges ($p = 2 > 1$), it follows by Comparison that $\sum \frac{|\sin(k)|}{k^2}$ converges. $\sum \frac{\sin(k)}{k^2}$ $\boxed{\text{converges absolutely}}$

e.) $\sum_{k=2}^{\infty} \frac{(-1)^k k}{k+1}$

$$\lim_{k\to\infty} a_k = \lim_{k\to\infty} (-1)^k \frac{k}{k+1} = \left(\lim_{k\to\infty} (-1)^k\right)\underbrace{\left(\lim_{k\to\infty} \frac{k}{k+1}\right)}_{=1} = \lim_{k\to\infty} (-1)^k \neq 0$$

Diverges by Divergence Test