§9.1: Sequences and Infinite Series

1.] Using the explicit formulas below, write out the first four terms of each sequence:

a.) $a_n = \dfrac{n}{n+1}$, for $n = 1, 2, 3\ldots$

$\left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots\right\}$

b.) $b_n = \dfrac{(-1)^n(n+1)}{3^n}$, for $n = 0, 1, 2, \ldots$

$\left\{1, -\frac{2}{3}, \frac{1}{3}, -\frac{4}{27}, \ldots\right\}$

2.] For each infinite sequence below: find the next two terms, determine a recurrence relation, and determine an explicit formula that describes the sequence:

a.) $\{a_n\} = \{3, 6, 12, 24, 48, \ldots\}$ 96, 192

$= \{3\cdot 2^0, 3\cdot 2^1, 3\cdot 2^2, 3\cdot 2^3, \ldots\}$

$\boxed{a_n = 3\cdot 2^n \quad n = 0, 1, 2, 3, \ldots}$

b.) $\{b_n\} = \{1, 0, 1, 0, 1, \ldots\}$ 0, 1

$\boxed{b_n = \dfrac{1 + (-1)^n}{2} \quad n = 0, 1, 2, \ldots}$

or $\boxed{b_n = \cos^2\left(\dfrac{n\pi}{2}\right) \quad n = 0, 1, 2, \ldots}$

3.] For each infinite sequence below, determine if the sequence converges or diverges to a limiting value, i.e., compute $\lim_{n\to\infty} a_n$.

a.) $\left\{\dfrac{(-1)^n}{n^2+1}\right\}_{n=1}^{\infty}$ $= \left\{-\frac{1}{2}, \frac{1}{5}, -\frac{1}{10}, \frac{1}{17}, \ldots\right\}$

$\lim_{n\to\infty} \dfrac{-1}{n^2+1} \le \lim_{n\to\infty} \dfrac{(-1)^n}{n^2+1} \le \lim_{n\to\infty} \dfrac{1}{n^2+1} \Rightarrow 0 \le \lim_{n\to\infty} \dfrac{(-1)^n}{n^2+1} \le 0$

$\Rightarrow \lim_{n\to\infty} \dfrac{(-1)^n}{n^2+1} = \boxed{0}$ converges

b.) $\{\cos(n\pi)\}_{n=1}^{\infty}$ $= \{-1, 1, -1, 1, -1, \ldots\}$

$\lim_{n\to\infty} \cos(n\pi) = \pm 1$ (always oscillating between -1 & 1) $\boxed{\text{Divergent}}$

c.) $a_{n+1} = -2a_n$ for $n = 1, 2, 3, \ldots$, where $a_1 = 1$

$= \{1, -2, 4, -8, 16, -32, \ldots\}$ $\lim_{n\to\infty} (-2)^n = \pm\infty$ $\boxed{\text{Divergent}}$

$a_n = (-2)^n \quad n = 0, 1, 2, \ldots$

d.) $a_n = \dfrac{4n^3}{n^3+1}$ for $n = 1, 2, 3, \ldots$

$\lim_{n\to\infty} \dfrac{4n^3}{n^3+1} = \lim_{n\to\infty} \dfrac{\cancel{n^3}(4)}{\cancel{n^3}(1 + 1/n^3)} = \lim_{n\to\infty} \dfrac{4}{1 + 1/n^3} = \dfrac{4}{1} = \boxed{4}$

converges

4.] Consider the series given by

$$\sum_{k=1}^{\infty} \frac{1}{2^k} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = \boxed{1}$$

a.) Find the first four partial sums S_1, S_2, S_3, and S_4 of the series.

$$S_1 = \frac{1}{2} = 1 - \frac{1}{2} = 1 - \frac{1}{2^1}$$

$$S_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4} = 1 - \frac{1}{4} = 1 - \frac{1}{2^2}$$

$$S_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8} = 1 - \frac{1}{8} = 1 - \frac{1}{2^3}$$

$$S_4 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = \frac{15}{16} = 1 - \frac{1}{16} = 1 - \frac{1}{2^4}$$

b.) Find a formula for the n^{th} partial sum S_n of the infinite series.

$$S_n = 1 - \frac{1}{2^n}$$

c.) Make a conjecture for the value of the infinite series, and determine if it converges or diverges.

$$\sum_{k=1}^{\infty} \frac{1}{2^k} = \lim_{n\to\infty} S_n = \lim_{n\to\infty} 1 - \frac{1}{2^n} = 1 - 0 = \boxed{1}$$

5.] Consider the series given by

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \cdots = \boxed{1}$$

a.) Find the first four partial sums S_1, S_2, S_3, and S_4 of the series.

$$S_1 = \frac{1}{2} = \frac{1}{1+1}$$

$$S_2 = \frac{1}{2} + \frac{1}{6} = \frac{2}{3} = \frac{2}{2+1}$$

$$S_3 = \frac{2}{3} + \frac{1}{12} = \frac{3}{4} = \frac{3}{3+1}$$

$$S_4 = \frac{3}{4} + \frac{1}{20} = \frac{4}{5} = \frac{4}{4+1}$$

b.) Find a formula for the n^{th} partial sum S_n of the infinite series.

$$S_n = \frac{n}{n+1}$$

c.) Make a conjecture for the value of the infinite series, and determine if it converges or diverges.

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{n}{n(1+\frac{1}{n})}$$

$$= \lim_{n\to\infty} \frac{1}{1+1/n} = \boxed{1}$$