§9.2: Sequences

1.] Determine the limits of the following sequences:

a.) $\left\{\dfrac{3n^3-1}{2n^3+1}\right\} = \left\{0, \frac{23}{17}, \frac{80}{55}, \dots\right\}$

$$\lim_{n\to\infty} \frac{3n^3-1}{2n^3+1} = \lim_{n\to\infty} \frac{3-1/n^3}{2+1/n^3} = \boxed{\frac{3}{2}}$$

Converges

b.) $\{\ln(n^3+1) - \ln(3n^3+10n)\}$

$$\lim_{n\to\infty} \ln(n^3+1) - \ln(3n^3+10n) = \lim_{n\to\infty} \ln\left(\frac{n^3+1}{3n^3+10n}\right) = \lim_{n\to\infty} \ln\left(\frac{1+1/n^3}{3+10/n^2}\right) = \boxed{\ln\left(\frac{1}{3}\right)}$$

Converges

c.) $\left\{\dfrac{(n+1)!}{n!}\right\} = \left\{\frac{2!}{1!}, \frac{3!}{2!}, \frac{4!}{3!}, \dots\right\} = \{2, 3, 4, \dots\}$

$$\lim_{n\to\infty} \frac{(n+1)!}{n!} = \lim_{n\to\infty} \frac{(n+1)n!}{n!} = \lim_{n\to\infty} (n+1) = \boxed{\infty}$$ Diverges

d.) $\left\{\left(1+\frac{2}{n}\right)^n\right\} = \left\{3, 9, \frac{125}{27}, \dots\right\}$ $\lim_{n\to\infty}\left(1+\frac{2}{n}\right)^n = \boxed{e^2}$ Converges

$$\lim_{x\to\infty}\left(1+\frac{2}{x}\right)^x$$
$$= \lim_{x\to\infty} e^{\ln\left(1+\frac{2}{x}\right)^x}$$
$$= \lim_{x\to\infty} e^{x\ln\left(1+\frac{2}{x}\right)}$$
$$= e^2$$

$$\lim_{x\to\infty} x\ln\left(1+\frac{2}{x}\right)$$
$$= \lim_{x\to\infty} \frac{\ln\left(1+\frac{2}{x}\right)}{\frac{1}{x}}$$
L'H
$$= \lim_{x\to\infty} \frac{\frac{1}{1+\frac{2}{x}}\cdot\left(-\frac{2}{x^2}\right)}{-\frac{1}{x^2}}$$
$$= \lim_{x\to\infty} \frac{2}{1+\frac{2}{x}}$$
$$= 2$$

2.] Consider the following Geometric sequences. Discuss the behavior of each and determine which sequences converge.

a.) $\{0.75^n\}$ b.) $\{(-0.75)^n\}$ c.) $\{1.16^n\}$ d.) $\{(-1.16)^n\}$

$r = 0.75$ $r = -0.75$ $r = 1.16$ $r = -1.16$

Converges, $|r|<1$ Converges, $|r|<1$ Diverges, $r>1$ Diverges, $r<-1.16$

3.] Prove that the following sequence converges by showing that the sequence is bounded and monotonic:

$$a_n = ne^{-n}, \quad \text{for } n = 1, 2, 3, \ldots$$

$\{a_n\} = \{\frac{1}{e}, \frac{2}{e^2}, \frac{3}{e^3}, \ldots\}$

Let $f(x) = xe^{-x}$, then

$f'(x) = e^{-x} - xe^{-x} = (1-x)e^{-x}$

Here, $f'(x) < 0$ for $x > 1$, thus $f(x)$ is decreasing for $x > 1$.

Hence, a_n is decreasing/monotonic.

• Every term is less than one, hence a_n is bounded above by 1.

• Therefore, $\{a_n\}$ is convergent.

$\lim_{x\to\infty} xe^{-x} = \lim_{x\to\infty} \frac{x}{e^x} = \lim_{x\to\infty} \frac{1}{e^x} = \boxed{0}$

a_n converges to zero!

4.] Compare the growth rates of sequences to determine whether the following sequences converge.

a.) $\left\{\frac{\ln(n^{10})}{0.00001n}\right\}$ $\lim_{n\to\infty} \frac{\ln(n^{10})}{.00001n} = \lim_{n\to\infty} \frac{10\ln(n)}{.00001n} = \frac{10}{.0001} \lim_{n\to\infty} \frac{\ln(n)}{n} = \boxed{0}$

since $\ln(n) \ll n$.

b.) $\left\{\frac{n^8 \ln(n)}{n^{8.001}}\right\}$ $\lim_{n\to\infty} \frac{n^8 \ln(n)}{n^{8.001}} = \lim_{n\to\infty} \frac{\ln(n)}{n^{.001}} = \boxed{0}$ since $\ln(n) \ll n^{.001}$

c.) $\left\{\frac{n!}{10^n}\right\}$ $\lim_{n\to\infty} \frac{n!}{10^n} = \boxed{\infty}$ since $n! \gg 10^n$.