§9.3: INFINITE SERIES

1.] Evaluate the following geometric sums:

a.) $\sum_{k=0}^{7} 2^k = \frac{a(1-r^8)}{1-r} = \frac{(1)(1-2^8)}{1-2} = \frac{-255}{-1} = \boxed{255}$

$a=1,\ r=2$

b.) $\sum_{k=3}^{10} \frac{3}{2^k} = \sum_{k=3}^{10} 3\left(\frac{1}{2}\right)^k = \sum_{j=0}^{7} 3\left(\frac{1}{2}\right)^{j+3} = \sum_{j=0}^{7} 3\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^j = \sum_{j=0}^{7}\left(\frac{3}{8}\right)\left(\frac{1}{2}\right)^j = \frac{\left(\frac{3}{8}\right)\left(1-\left(\frac{1}{2}\right)^8\right)}{1-\frac{1}{2}}$

$j=k-3,\ k=j+3$

$k=3 \to j=0,\ k=10 \to j=7$

$a=\frac{3}{8},\ r=\frac{1}{2}$ $= \boxed{\frac{765}{1024}}$

2.] Evaluate the following geometric series:

a.) $\sum_{k=0}^{\infty} 1.1^k \longrightarrow \boxed{\text{Diverges}}$

$r=1.1>1$

b.) $\sum_{k=0}^{\infty} e^{-k} = \sum_{k=0}^{\infty}\left(\frac{1}{e}\right)^k = \frac{a}{1-r} = \frac{1}{1-\frac{1}{e}} = \frac{1}{\frac{e-1}{e}} = \boxed{\frac{e}{e-1}}$ Converges

$a=1,\ r=\frac{1}{e}$

c.) $\sum_{k=2}^{\infty} 3(-0.75)^k = \sum_{j=0}^{\infty} 3(-0.75)^{j+2} = \sum_{j=0}^{\infty} 3(-0.75)^2(-0.75)^j = \sum_{j=0}^{\infty}\left(\frac{27}{16}\right)\left(-\frac{3}{4}\right)^j = \frac{\frac{27}{16}}{1-\left(-\frac{3}{4}\right)}$

$j=k-2,\ k=j+2$

$k=2 \to j=0,\ k=\infty \to j=\infty$

$a=\frac{27}{16},\ r=-\frac{3}{4}$ $= \boxed{\frac{27}{28}}$

d.) $\sum_{k=1}^{\infty} 2^{-3k} = \sum_{k=1}^{\infty}(2^{-3})^k = \sum_{k=1}^{\infty}\left(\frac{1}{8}\right)^k = \sum_{j=0}^{\infty}\left(\frac{1}{8}\right)^{j+1} = \sum_{j=0}^{\infty}\left(\frac{1}{8}\right)\left(\frac{1}{8}\right)^j = \frac{\frac{1}{8}}{1-\frac{1}{8}}$

$j=k-1,\ k=j+1$

$k=1 \to j=0$

$= \boxed{7}$

e.) $\frac{1}{16} + \frac{3}{64} + \frac{9}{256} + \frac{27}{1024} + \cdots$

$= \frac{1}{16}\left(1 + \frac{3}{4} + \frac{9}{16} + \frac{27}{64} + \cdots\right)$

$= \sum_{k=0}^{\infty}\left(\frac{1}{16}\right)\left(\frac{3}{4}\right)^k = \frac{1/16}{1-3/4} = \frac{1/16}{1/4} = \boxed{\frac{1}{4}}$

3.] Evaluate the following telescoping series:

a.) $\sum_{k=1}^{\infty}\left(\frac{1}{k+1}-\frac{1}{k+2}\right) = \left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\left(\frac{1}{4}-\frac{1}{5}\right)+\left(\frac{1}{5}-\frac{1}{6}\right)+\cdots = \boxed{\frac{1}{2}}$

$S_1 = \left(\frac{1}{2}-\frac{1}{3}\right)$

$S_2 = \left(\frac{1}{2}-\cancel{\frac{1}{3}}\right)+\left(\cancel{\frac{1}{3}}-\frac{1}{4}\right) = \frac{1}{2}-\frac{1}{4}$

$S_3 = \left(\frac{1}{2}-\frac{1}{4}\right)+\left(\frac{1}{4}-\frac{1}{5}\right) = \frac{1}{2}-\frac{1}{5}$

$S_4 = \left(\frac{1}{2}-\frac{1}{5}\right)+\left(\frac{1}{5}-\frac{1}{6}\right) = \frac{1}{2}-\frac{1}{6}$

$S_n = \frac{1}{2}-\frac{1}{n+2}$

$\sum_{k=1}^{\infty}\left(\frac{1}{k+1}-\frac{1}{k+2}\right) = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{1}{2}-\cancel{\frac{1}{n+2}} = \boxed{\frac{1}{2}}$

Partial Fractions

b.) $\sum_{k=1}^{\infty}\frac{6}{k^2+2k} = \sum_{k=1}^{\infty}\left(\frac{3}{k}-\frac{3}{k+2}\right) = \left(\frac{3}{1}-\frac{3}{3}\right)+\left(\frac{3}{2}-\frac{3}{4}\right)+\left(\frac{3}{3}-\frac{3}{5}\right)+\left(\frac{3}{4}-\frac{3}{6}\right)+\cdots = \boxed{\frac{9}{2}}$

$S_1 = \left(\frac{3}{1}-\frac{3}{3}\right)$

$S_2 = \left(\frac{3}{1}-\frac{3}{3}\right)+\left(\frac{3}{2}-\frac{3}{4}\right) = \frac{9}{2}-\frac{3}{3}-\frac{3}{4}$

$S_3 = \left(\frac{3}{1}-\cancel{\frac{3}{3}}\right)+\left(\frac{3}{2}-\frac{3}{4}\right)+\left(\cancel{\frac{3}{3}}-\frac{3}{5}\right) = \frac{9}{2}-\frac{3}{4}-\frac{3}{5}$

$S_4 = \frac{9}{2}-\cancel{\frac{3}{4}}-\frac{3}{5}+\left(\cancel{\frac{3}{4}}-\frac{3}{6}\right) = \frac{9}{2}-\frac{3}{5}-\frac{3}{6}$

$S_5 = \frac{9}{2}-\frac{3}{5}-\frac{3}{6}+\left(\frac{3}{5}-\frac{3}{7}\right) = \frac{9}{2}-\frac{3}{6}-\frac{3}{7}$

$S_n = \frac{9}{2}-\frac{3}{n+1}-\frac{3}{n+2}$

$\lim_{n\to\infty} S_n = \boxed{\frac{9}{2}}$

4.] Evaluate the following series:

a.) $\sum_{k=0}^{\infty} 3\left(\frac{2}{5}\right)^k - 2\left(\frac{5}{7}\right)^k$

$= \sum_{k=0}^{\infty} 3\left(\frac{2}{5}\right)^k - \sum_{k=0}^{\infty} 2\left(\frac{5}{7}\right)^k$

$= \frac{3}{1-2/5} - \frac{2}{1-5/7} = \frac{3}{3/5} - \frac{2}{2/7} = 5-7 = \boxed{-2}$

b.) $\sum_{k=1}^{\infty}\left(\arcsin\left(\frac{1}{k}\right)-\arcsin\left(\frac{1}{k+1}\right)\right) =$

$S_1 = \arcsin(1) - \arcsin\left(\frac{1}{2}\right)$

$S_2 = \left(\arcsin(1) - \cancel{\arcsin\left(\frac{1}{2}\right)}\right) + \left(\cancel{\arcsin\left(\frac{1}{2}\right)} - \arcsin\left(\frac{1}{3}\right)\right) = \arcsin(1) - \arcsin\left(\frac{1}{3}\right)$

$S_2 = \left(\arcsin(1) - \cancel{\arcsin\left(\frac{1}{3}\right)}\right) + \left(\cancel{\arcsin\left(\frac{1}{3}\right)} - \arcsin\left(\frac{1}{4}\right)\right) = \arcsin(1) - \arcsin\left(\frac{1}{4}\right)$

$\vdots$

$S_n = \arcsin(1) - \arcsin\left(\frac{1}{n+1}\right)$

$\lim_{n\to\infty} \arcsin(1) - \arcsin\left(\frac{1}{n+1}\right)$

$= \arcsin(1) - \arcsin(0) = \boxed{\frac{\pi}{2}}$