§5.1-5.4: Review of Integration

1.] Find the derivative of the following functions:

a.) $A(x) = \int_0^x t^2\, dt$ b.) $B(x) = \int_1^{x^4} \frac{1}{t^2}\, dt$

$A'(x) = \boxed{x^2}$

$B'(x) = \frac{1}{(x^4)^2} \cdot 4x^3 = \boxed{\frac{4}{x^5}}$

2.] Suppose $f'(x) = 4x^3 + 1$ and $f(1) = 2$. Find $f(x)$.

$f'(x) = 4x^3 + 1$

$\Rightarrow \int f'(x)dx = \int 4x^3 + 1 dx$

$f(x) = x^4 + x + C$

$\Rightarrow f(1) = 1^4 + 1 + C$

$\Rightarrow 2 = 2 + C$

$\Rightarrow C = 0$

$\boxed{f(x) = x^4 + x}$

3.] Compute the following definite integrals:

a.) $\int_0^2 5x^4\, dx = x^5\Big|_0^2 = 2^5 - 0^5 = \boxed{32}$

b.) $\int_0^1 (x + \sqrt{x})\, dx = \int_0^1 x + x^{1/2} dx = \frac{1}{2}x^2 + \frac{1}{3/2}x^{3/2}\Big|_0^1 = \frac{1}{2} + \frac{2}{3} = \boxed{\frac{7}{6}}$

c.) $\int_0^{\ln 8} e^x\, dx = e^x\Big|_0^{\ln(8)} = e^{\ln(8)} - e^0 = 8 - 1 = \boxed{7}$

d.) $\int_{\pi/4}^{\pi/2} 8\csc^2(x)\, dx = 8(-\cot(x))\Big|_{\pi/4}^{\pi/2}$

$= -8\cot(\pi/2) - (-8\cot(\pi/4))$

$= -8(0) + 8(1)$

$= \boxed{8}$

4.] Evaluate the following definite integrals. Use symmetry where necessary.

a.) $\int_{-2}^{2} (x^9 - 3x^5 + 3x^2 - 10)\, dx$

$$= \int_{-2}^{2} \underbrace{x^9 - 3x^5}_{\text{odd}}\, dx + \int_{-2}^{2} \underbrace{3x^2 - 10}_{\text{even}}\, dx = 0 + 2\int_0^2 3x^2 - 10\, dx$$

$$= 2(x^3 - 10x)\Big|_0^2$$

$$= 2(2^3 - 20) = 2(8-20) = \boxed{-24}$$

b.) $\int_{-\pi/2}^{\pi/2} \left(\underbrace{\cos(x)}_{\text{even}} + \underbrace{\overset{(-)}{x}\,\overset{(+)}{\cos(x)}\,\overset{(+)}{\sin^2(x)}}_{\text{odd}} \right) dx$

$$= 2\int_0^{\pi/2} \cos(x)\, dx + \int_{-\pi/2}^{\pi/2} x\cos(x)\sin^2(x)\, dx = 2\left(\sin(x)\right)\Big|_0^{\pi/2} + 0$$

$$= \boxed{2}$$

c.) $\int_{-1}^{1} (\underbrace{1 - |x|}_{\text{even}})\, dx$

$$= 2\int_0^1 (1 - |x|)\, dx = 2\int_0^1 (1-x)\, dx = 2\left(x - \tfrac{1}{2}x^2\right)\Big|_0^1 = 2\left(1 - \tfrac{1}{2}\right) = \boxed{1}$$

note: on $[0,1]$, $|x| = x$

5.] Find the average value of the function $f(x) = x^2 + 1$ over the interval $[-1, 3]$.

$$\bar{f} = \frac{1}{b-a}\int_a^b f(x)\, dx$$

$$\bar{f} = \frac{1}{3-(-1)}\int_{-1}^{3} x^2 + 1\, dx = \frac{1}{4}\left(\frac{1}{3}x^3 + x\right)\Big|_{-1}^{3} = \frac{1}{4}(9+3) - \frac{1}{4}\left(-\frac{1}{3} - 1\right) = 3 - \frac{1}{4}\left(-\frac{4}{3}\right)$$

$$= \boxed{\frac{10}{3}}$$

6.] Find the average value of the function $f(x) = \dfrac{1}{1+x^2}$ over the interval $[-1, 1]$.

$$\bar{f} = \frac{1}{1-(-1)}\int_{-1}^{1} \frac{1}{1+x^2}\, dx$$

$$= \frac{1}{2} \cdot 2\int_0^1 \frac{1}{1+x^2}\, dx = \arctan(x)\Big|_0^1 = \arctan(1) - \arctan(0)$$

$$= \pi/4 - 0$$

$$= \boxed{\pi/4}$$