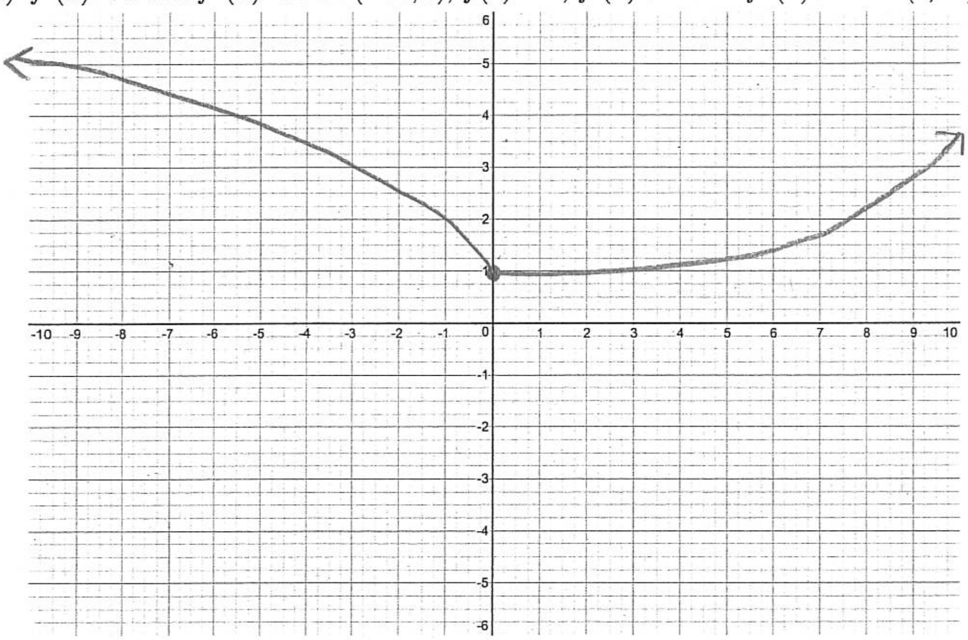
§4.2 (PART 2): CONCAVITY AND SECOND DERIVATIVE TEST

1.] Let $f(x)$ be a continuous function on $(-\infty, \infty)$. Sketch a graph of f that satisfies the following conditions:

a.) $f'(x) < 0$ and $f''(x) < 0$ on $(-\infty, 0)$; $f(0) = 1$; $f'(x) > 0$ and $f''(x) > 0$ on $(0, \infty)$.

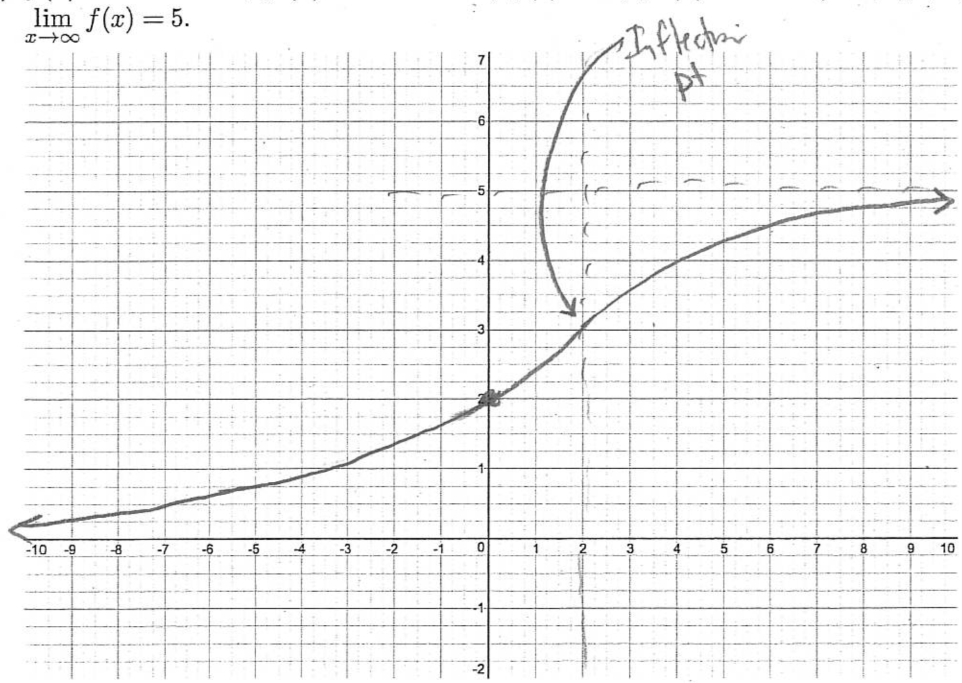


$(-\infty, 0)$:

$f(0) = 1 \Rightarrow (0,1)$

$(0, \infty)$:

b.) $f(x) > 0$ for all x; $f'(x) > 0$ for all x; $f(0) = 2$; $f''(x) > 0$ on $(-\infty, 2)$; $f''(x) < 0$ on $(2, \infty)$; $\lim_{x\to\infty} f(x) = 5$.



- $f(x) > 0$ for all x. (always above x-axis)
- $f'(x) > 0$ for all x (always increasing)
- $f(0) = 2$ $(0,2)$
- CU on $(-\infty, 2)$
- CD on $(2, \infty)$
- HA at $y = 5$

2.] Determine the intervals on which the function $f(x) = 3x^5 - 30x^4 + 80x^3 + 100$ are concave up or concave down. Identify all inflection points.

$f'(x) = 15x^4 - 120x^3 + 240x^2$

$\Rightarrow f''(x) = 60x^3 - 360x^2 + 480x$

$\Rightarrow f''(x) = 60x(x^2 - 6x + 8)$

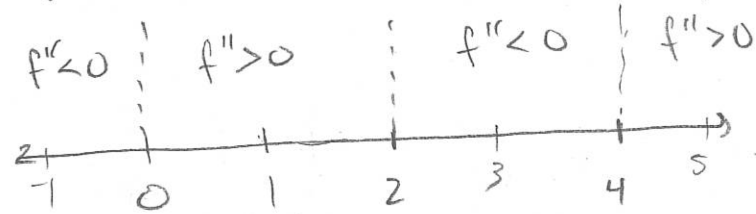
$\Rightarrow f''(x) = 60x(x-2)(x-4)$

$\Rightarrow 0 = 60x(x-2)(x-4)$

Candidate Inflection points:

$x = 0, 2, 4$

Sign Analysis: $f''(x) = 60x(x-2)(x-4)$



- f is concave up on $(0,2) \cup (4,\infty)$
- f is concave down on $(-\infty,0) \cup (2,4)$
- $x=0$, $x=2$, $x=4$ are inflection points

3.] Locate the critical points of the function $f(x) = x^2 e^{-x}$. Then use the Second Derivative Test to determine whether the critical points correspond to local maxima, local minima, or neither.

$f(x) = x^2 e^{-x}$

$f'(x) = (2x)(e^{-x}) + (x^2)(-e^{-x})$

$\Rightarrow f'(x) = 2xe^{-x} - x^2 e^{-x}$

$\Rightarrow f'(x) = e^{-x} x(2-x)$

$\Rightarrow e^{-x} x(2-x) = 0$

Never zero!

$x=0,\ x=2$

Critical points: $\boxed{c_1 = 0,\ c_2 = 2}$

$f''(x) = 2e^{-x} - 2xe^{-x} - 2xe^{-x} - x^2(-e^{-x})$

$f''(x) = 2e^{-x} - 4xe^{-x} + x^2 e^{-x}$

$f''(x) = e^{-x}(x^2 - 4x + 2)$

Second Derivative Test:

$f''(c_1) = f''(0) = e^{-0}(0^2 - 4(0) + 2) = 2 > 0$

Since $f''(0) > 0$, there is a local min at $x=0$

$f''(c_2) = f''(2) = e^{-2}(2^2 - 4(2) + 2) = e^{-2}(-2) < 0$

Since $f''(2) < 0$, there is a local max at $x=2$