§4.4: Optimization

1.] What two positive real numbers with a product of of 50 have the smallest possible sum?

Objective: Minimize Sum: $S = x + y \longrightarrow S(x) = x + \frac{50}{x}$

Constraint: $x > 0, y > 0, xy = 50$

$\Rightarrow y = \frac{50}{x}$

$\Rightarrow S'(x) = 1 - \frac{50}{x^2}$

$\Rightarrow 0 = 1 - \frac{50}{x^2}$

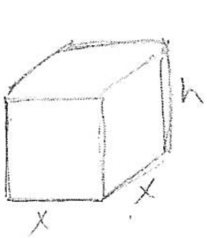
$\Rightarrow \frac{50}{x^2} = 1$

$\Rightarrow x^2 = 50$

$\Rightarrow x = \pm\sqrt{50}$

$\Rightarrow \boxed{x = 5\sqrt{2}}$

2.] Of all boxes with a square base and a volume of 100 m^3, which one has the smallest minimum surface area?



Objective: Minimize Surface Area: $S_a = 2x^2 + 4xh$

Constraint: $x > 0, y > 0, x^2 h = 100$

$\Rightarrow h = \frac{100}{x^2}$

$\hookrightarrow S_a(x) = 2x^2 + 4x\left(\frac{100}{x^2}\right)$

$\Rightarrow S_a(x) = 2x^2 + \frac{400}{x}$

$\Rightarrow S_a'(x) = 4x - \frac{400}{x^2}$

$\Rightarrow 0 = 4x - \frac{400}{x^2}$

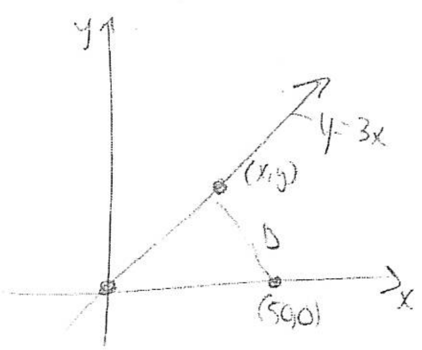
$\Rightarrow x^3 = 100$

$\Rightarrow \boxed{x = \sqrt[3]{100}}$

$\Rightarrow h = \frac{100}{(\sqrt[3]{100})^2}$

$\Rightarrow \boxed{h = \sqrt[3]{100}}$

3.] Find the point on the line $y = 3x$ that is the closest point to $(50, 0)$. What is the least distance from this point to $(50, 0)$?



Objective: Min Distance: $D = \sqrt{(x-50)^2 + y^2}$

*we need only minimize the expression under the root!

Constraint: $x, y > 0$, $y = 3x$

$D(x) = (x-50)^2 + (3x)^2$

$\Rightarrow D(x) = x^2 - 100x + 2500 + 9x^2$

$\Rightarrow D(x) = 10x^2 - 100x + 2500$

$\Rightarrow D'(x) = 20x - 100$

$\Rightarrow 0 = 20x - 100$

$\Rightarrow \boxed{x = 5}$

$\Rightarrow \boxed{y = 15}$

$D''(x) = 20 > 0$

$\Rightarrow$ minimum!

Min Distance: $D = \sqrt{(5-50)^2 + (15)^2}$

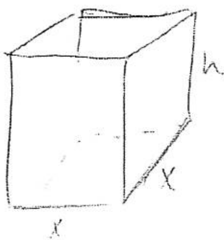
$= \sqrt{45^2 + 15^2}$

$= \sqrt{4(15)^2}$

$= \boxed{30}$

4.] You work for a company that makes jewelry boxes. Your boss tells you that each jewelry box must have a square base and an open top and that you can spend \$3.75 on the materials for each box. The people in production tell you that the material for the sides of the box costs 2 cents per square inch while the reinforced material for the base of the box costs 5 cents per square inch. What is the largest volume jewelry box that you can make and still stay within budget?

Objective: Max Volume: $V = x^2 h$



Constraint: Stay in Budget: $0.05x^2 + 0.02(4xh) = 3.75$

$\Rightarrow 0.05x^2 + 0.08xh = 3.75$

$\Rightarrow 0.08xh = 3.75 - 0.05x^2$

$\Rightarrow h = \frac{3.75}{0.08x} - \frac{0.05}{0.08}x$

$\Rightarrow h = \frac{46.875}{x} - 0.625x$

$V(x) = x^2\left(\frac{46.875}{x} - 0.625x\right)$

$\Rightarrow V(x) = 46.875x - 0.625x^3$

$\Rightarrow V'(x) = 46.875 - 0.625(3)x^2$

$\Rightarrow 0 = 46.875 - 1.875x^2$

$\Rightarrow x^2 = 25$

$\Rightarrow \boxed{x = 5}$

$\Rightarrow h = \frac{46.875}{5} - 0.625(5)$

$\Rightarrow \boxed{h = 6.25}$

Max Volume $= (5)^2(6.25)$

$= \boxed{156.25 \text{ in}^3}$

5.] A length of wire of length 60 ft is cut, and the resulting two pieces are formed to make a circle and a square. Where should the wire be cut to (a) minimize and (b) maximize the combined area of the circle and square?

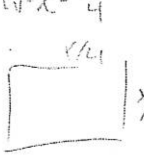
Objective: Min/Max Total Area: $A = \frac{x^2}{16} + \frac{1}{4\pi}(60-x)^2$

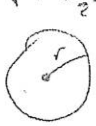
Constraint: $0 \leq x \leq 60$

$w = l = \frac{x}{4}$

$2\pi r = 60 - x$

$r = \frac{60-x}{2\pi}$





$A_s = \frac{x}{4}\cdot\frac{x}{4}$

$A_s = \frac{x^2}{16}$

$A_c = \pi\left(\frac{60-x}{2\pi}\right)^2$

$A_c = \frac{1}{4\pi}(60-x)^2$

b) Maximum must occur at endpoints:

$A(0) = \frac{1}{4\pi}(60)^2 = 286.48$

$A(60) = \frac{1}{16}(60)^2 = 225$

Hence, $x = 0$ yields maximum area.

$\Rightarrow A(x) = \frac{1}{16}x^2 + \frac{1}{4\pi}(60-x)^2$

$\Rightarrow A'(x) = \frac{1}{8}x + \frac{1}{4\pi}2(60-x)(-1)$

$\Rightarrow A'(x) = \frac{1}{8}x + \frac{x}{2\pi} - \frac{30}{\pi}$

$\Rightarrow 0 = \frac{1}{8}x + \frac{x}{2\pi} - \frac{30}{\pi}$

$\Rightarrow \frac{30}{\pi} = x\left(\frac{1}{8} + \frac{1}{2\pi}\right)$

$\Rightarrow x = \frac{30}{\pi}\left(\frac{16\pi}{8+2\pi}\right) \approx \boxed{33.61 \text{ ft}}$

Solves part a.

$\Rightarrow A''(x) = \frac{1}{8} + \frac{1}{2\pi} > 0$

Minimum