§4.6: Mean Value Theorem

1.] Verify that $f(x) = x^3 - x^2 - 6x + 2$ satisfies the hypothesis of Rolle's theorem for the interval $[0, 3]$ and then find all c that satisfy the conclusion.

1.) f is cont. on $[0,3]$ ✓

2.) $f'(x) = 3x^2 - 2x - 6$

f is diff on $(0,3)$ ✓

3.) $f(0) = 0^3 - 0^2 - 6(0) + 2 = 2$

$f(3) = 3^3 - 3^2 - 6(3) + 2 = 2$ ✓

Hence, by Rolle's Theorem, there exists a value $c \in (0,3)$ such that

$f'(c) = 0$

$\Rightarrow 3c^2 - 2c - 6 = 0$

$\Rightarrow c = \frac{-(-2)}{2(3)} \pm \frac{\sqrt{(-2)^2 - 4(3)(-6)}}{2(3)}$

$\Rightarrow c = \frac{1}{3} \pm \frac{\sqrt{76}}{6}$

$c = \frac{1}{3} \pm \frac{\sqrt{19}}{3}$

$\boxed{c = \frac{1}{3} + \frac{\sqrt{19}}{3}}$

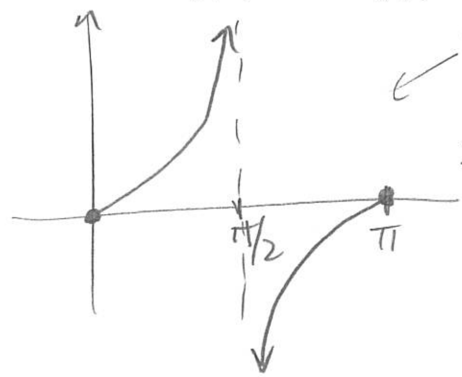
2.] Let $f(x) = \tan(x)$. Show that $f(0) = f(\pi)$ but that there is no value $c \in (0, \pi)$ such that $f'(c) = 0$. Why does this not contradict Rolle's theorem?

1.) f is not cont. on $[0,\pi]$ ✗

2.) f is not diff on $(0,\pi)$ ✗

3.) $f(0) = \tan(0) = 0$ ✓

$f(\pi) = \tan(\pi) = 0$



There is no value $c \in (0,\pi)$ such that $f'(c) = 0$.

This function doesn't contradict the theorem because it doesn't satisfy hypothesis.

3.] Verify that $f(x) = x^3 - 3x + 2$ satisfies the hypotheses of the Mean Value Theorem on $[-2, 2]$ and then find all c that satisfy the conclusion.

1.) f is cont on $[-2,2]$ ✓

2.) $f'(x) = 3x^2 - 3$

f is diff on $(-2,2)$ ✓

Hence by MVT, there exists a value $c \in (-2,2)$ such that

$f'(c) = \frac{f(b) - f(a)}{b - a}$

$\Rightarrow f'(c) = \frac{f(2) - f(-2)}{2 - (-2)}$

$3c^2 - 3 = \frac{[2^3 - 3(2) + 2] - [(-2)^3 - 3(-2) + 2]}{4}$

$\Rightarrow 3c^2 - 3 = \frac{4 - 0}{4}$

$\Rightarrow 3c^2 - 3 = 1$

$\Rightarrow 3c^2 = 4$

$\Rightarrow c^2 = \frac{4}{3}$

$\Rightarrow c = \pm \frac{2}{\sqrt{3}}$

$\boxed{c_1 = \frac{2}{\sqrt{3}} \quad c_2 = -\frac{2}{\sqrt{3}}}$

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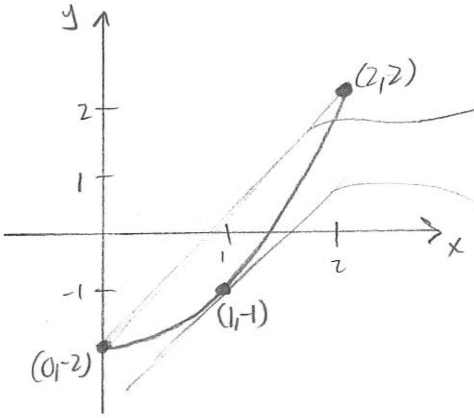
4.] Use the Mean Value Theorem to show there is some value $c \in (0, 2)$ at which the tangent line to the function $f(x) = x^2 - 2$ has slope 2. Use $f'(x)$ to find this value of c algebraically.



$m_{sec} = \frac{f(2)-f(0)}{2-0} = \frac{2-(-2)}{2-0} = \frac{4}{2} = 2$

$m_{tan} = f'(c)$

$f'(c) = 2$

$\Rightarrow 2c = 2$

$\Rightarrow \boxed{c = 1}$

This is the value guaranteed by the MVT.

5.] Law enforcement has been known to issue speeding tickets to drivers who pass between successive EZ pass booths in too short of a time interval. Assume EZ pass booths A and B are 100 miles apart. Use the mean value theorem to demonstrate that a driver who passes booth A at 1 PM and booth B at 2 PM was necessarily speeding at some time between the two booths. What assumptions are you making about the driver's position function?

m_{sec} = "Average velocity" $= \frac{100-0}{1-0} \left(\frac{\text{distance}}{\text{hr}}\right) = \frac{100}{1} = 100$

m_{tan} = "Instantaneous velocity"

Some time between Booth A and Booth B, the person must have been traveling 100 mph!

Note: we're assuming the person was driving a car, not teleporting. (i.e. function is continuous).

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