§3.7 (Part 2): Bifurcations in Linear Systems

1.] Consider the one-parameter family of of linear systems in the parameter a:

$$\frac{dy}{dt} = \begin{bmatrix} -2 & a \\ -2 & 0 \end{bmatrix} y.$$

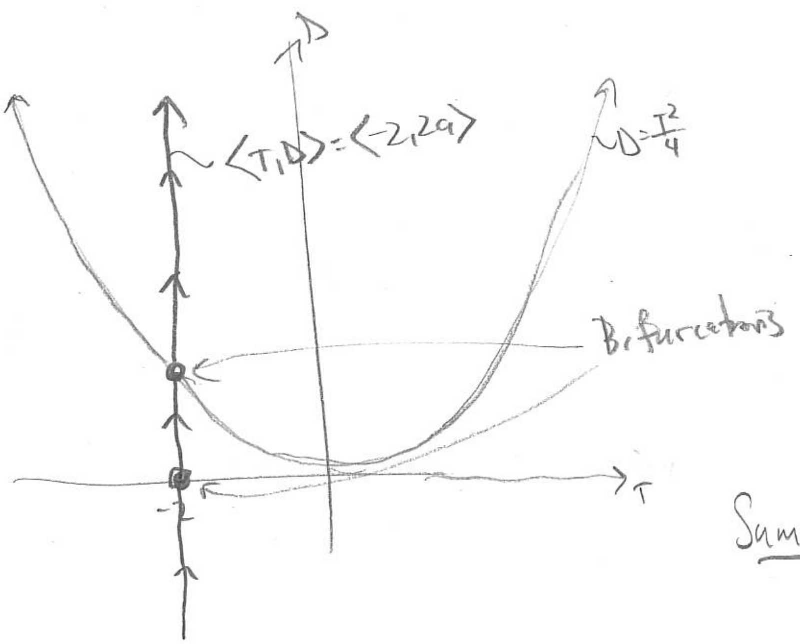
Determine the critical bifurcation values for a and discuss the type of system along with the stability that lies in each region between bifurcations.

Trace & Determinant:

tr(A): $T = -2 + 0 = -2$
det(A): $D = (-2)(0) - (-2)(a) = 2a$

$\Rightarrow \boxed{\langle T, D \rangle = \langle -2, 2a \rangle}$

As a increases, a vertical line at $T = -2$ is traversed from South to North.

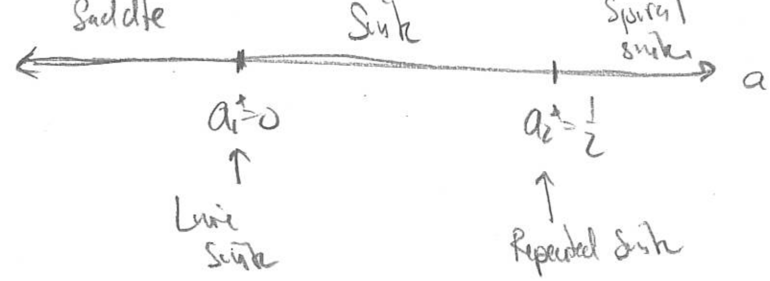


Bifurcations occur when:

1.) $D = 0 \Rightarrow 2a = 0 \Rightarrow \boxed{a_1^* = 0}$

2.) $D = T^2/4 \Rightarrow 2a = \frac{(-2)^2}{4} \Rightarrow 2a = 1$
$\Rightarrow \boxed{a_2^* = \frac{1}{2}}$

Summary:



The system is stable for $a \in [0, \infty)$

2.] Consider the one-parameter family of of linear systems in the parameter a:

$$\frac{dy}{dt} = \begin{bmatrix} 2a & -1 \\ 1 & -a \end{bmatrix} y.$$

Determine the critical bifurcation values for a and discuss the type of system along with the stability that lies in each region between bifurcations.

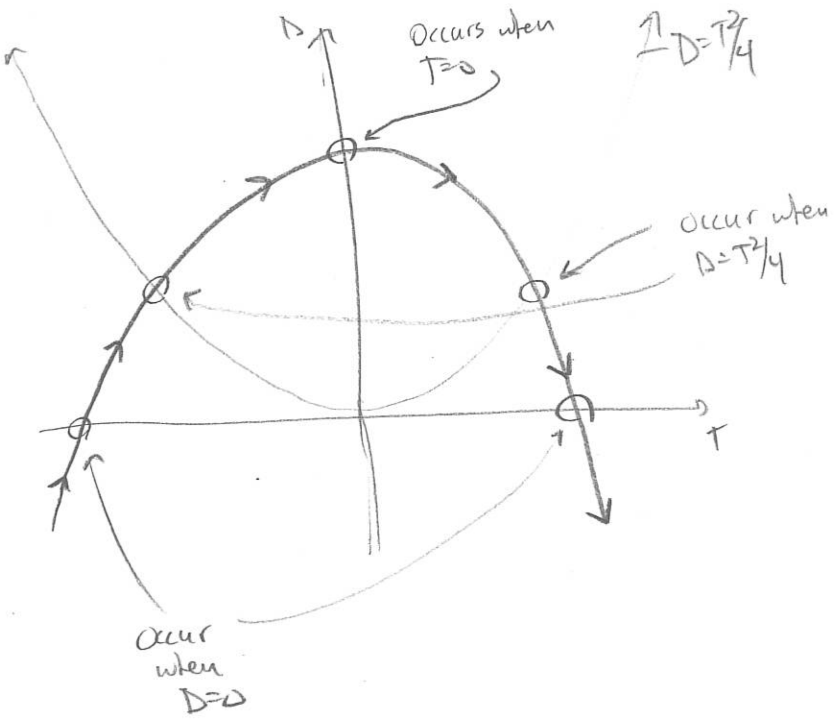
Trace & Determinant:

$tr(A)$: $T = 2a + (-a) = a$

$det(A)$: $D = (2a)(-a) - (-1)(1) = 1 - 2a^2$

$\langle T, D \rangle = \langle a, 1-2a^2 \rangle$

As a increases, a downward facing parabola is traversed.



Bifurcations:

1.) $D=0 \Rightarrow 1-2a^2=0 \Rightarrow a^2=\frac{1}{2}$

$\Rightarrow a = \pm\sqrt{\frac{1}{2}}$

$\boxed{a_1^* = -\frac{1}{\sqrt{2}}, \quad a_5^* = \frac{1}{\sqrt{2}}}$

2.) $D = T^2/4 \Rightarrow 1-2a^2 = \frac{a^2}{4}$

$\Rightarrow \frac{9a^2}{4} = 1$

$\Rightarrow a^2 = \frac{4}{9}$

$\Rightarrow a = \pm\frac{2}{3}$

$\boxed{a_2^* = -\frac{2}{3}, \quad a_4^* = \frac{2}{3}}$

3.) $T=0 \Rightarrow a=0 \Rightarrow \boxed{a_3^* = 0}$

Summary: The system is stable for $a \in [-\frac{1}{\sqrt{2}}, 0)$