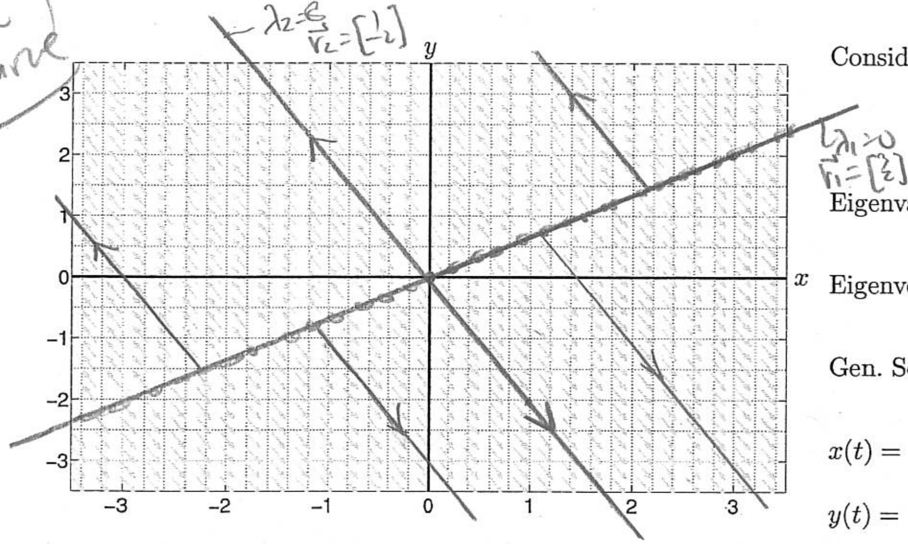
§3.5 (PART 2): STABILITY - ZERO EIGENVALUES

Line Source



Consider the system:

$$\frac{d\boldsymbol{y}}{dt} = \begin{bmatrix} 2 & -3 \\ -4 & 6 \end{bmatrix} \boldsymbol{y}$$

Eigenvalues: $\lambda_1 = 0,\ \lambda_2 = 8$

Eigenvectors: $\vec{v}_1 = \begin{bmatrix}3\\2\end{bmatrix},\ \vec{v}_2 = \begin{bmatrix}1\\-2\end{bmatrix}$

Gen. Sol. : $\vec{y}(t) = c_1\begin{bmatrix}3\\2\end{bmatrix} + c_2\begin{bmatrix}1\\-2\end{bmatrix}e^{8t}$

$x(t) =$ $3c_1 + c_2e^{8t}$

$y(t) =$ $2c_1 - 2c_2e^{8t}$

For the system above, find the unique solution if $\boldsymbol{y}(0) = \begin{bmatrix}1\\2\end{bmatrix}$ and describe the behavior of $x(t)$ and $y(t)$ as $t \to \infty$.

E-values: $\lambda^2 - tr(A)\lambda + det(A) = 0$

$\Rightarrow \lambda^2 - 8\lambda + 0 = 0$

$\Rightarrow \lambda(\lambda - 8) = 0$

$\boxed{\lambda_1 = 0,\ \lambda_2 = 8}$

E-vectors: $\boxed{\lambda_1 = 0}$ $\begin{cases} 2x - 3y = 0 \\ -4x + 6y = 0 \end{cases} \Rightarrow \begin{cases} y = \frac{2}{3}x \\ y = \frac{2}{3}x \end{cases} \Rightarrow \boxed{\vec{v}_1 = \begin{bmatrix}3\\2\end{bmatrix}}$

$\boxed{\lambda_2 = 8}$ $\begin{cases} (2-8)x - 3y = 0 \\ -4x + (6-8)y = 0 \end{cases} \Rightarrow \begin{cases} -6x - 3y = 0 \\ -4x - 2y = 0 \end{cases} \Rightarrow \begin{cases} y = -2x \\ y = -2x \end{cases}$ $\boxed{\vec{v}_2 = \begin{bmatrix}1\\-2\end{bmatrix}}$

Gen. Sol: $\vec{y}(t) = c_1\vec{v}_1e^{\lambda_1 t} + c_2\vec{v}_2e^{\lambda_2 t}$

$\Rightarrow \boxed{\vec{y}(t) = c_1\begin{bmatrix}3\\2\end{bmatrix} + c_2\begin{bmatrix}1\\-2\end{bmatrix}e^{8t}}$

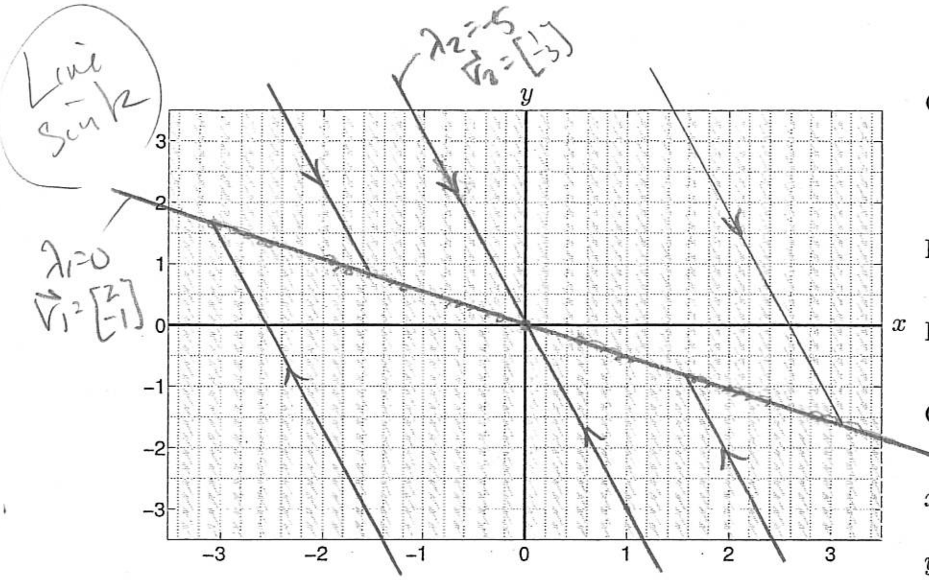
Unique Sol: $\vec{y}(0) = c_1\begin{bmatrix}3\\2\end{bmatrix} + c_2\begin{bmatrix}1\\-2\end{bmatrix}$

$\Rightarrow \begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}3c_1 + c_2\\2c_1 - 2c_2\end{bmatrix}$

$\Rightarrow \begin{matrix} 6c_1 + 2c_2 = 2 \\ 2c_1 - 2c_2 = 2 \end{matrix} \Rightarrow \begin{cases} 8c_1 = 4 \\ c_2 = 1 - 3c_1 \end{cases} \Rightarrow \begin{cases} c_1 = 1/2 \\ c_2 = -1/2 \end{cases}$

$$\boxed{\begin{aligned} \vec{y}(t) &= \frac{1}{2}\begin{bmatrix}3\\2\end{bmatrix} - \frac{1}{2}\begin{bmatrix}1\\-2\end{bmatrix}e^{8t} \\ x(t) &= \frac{3}{2} - \frac{1}{2}e^{8t} \\ y(t) &= 1 + e^{8t} \end{aligned}}$$

As $t \to \infty$,
$x(t) \to -\infty$
$y(t) \to \infty$



Consider the system:

$$\frac{d\boldsymbol{y}}{dt} = \begin{bmatrix} 1 & 2 \\ -3 & -6 \end{bmatrix} \boldsymbol{y}$$

Eigenvalues: $\lambda_1 = 0, \ \lambda_2 = -5$

Eigenvectors: $\vec{v}_1 = \begin{bmatrix}2\\-1\end{bmatrix} \quad \vec{v}_2 = \begin{bmatrix}1\\-3\end{bmatrix}$

Gen. Sol. : $\vec{y}(t) = C_1\begin{bmatrix}2\\-1\end{bmatrix} + C_2\begin{bmatrix}1\\-3\end{bmatrix}e^{-5t}$

$x(t) = 2C_1 + C_2 e^{-5t}$

$y(t) = -C_1 - 3C_2 e^{-5t}$

For the system above, find the unique solution if $\boldsymbol{y}(0) = \begin{bmatrix}7\\2\end{bmatrix}$ and describe the behavior of $x(t)$ and $y(t)$ as $t \to \infty$.

E-values: $\lambda^2 - tr(A)\lambda + det(A) = 0$

$\Rightarrow \lambda^2 + 5\lambda + 0 = 0$

$\Rightarrow \lambda(\lambda+5) = 0$

$\boxed{\lambda_1 = 0 \quad \lambda_2 = -5}$ Line Sink

E-vectors: $\boxed{\lambda_1 = 0}$ $\begin{cases} x + 2y = 0 \\ -3x - 6y = 0 \end{cases} \Rightarrow \begin{cases} y = -\frac{1}{2}x \\ y = -\frac{1}{2}x \end{cases} \to \boxed{\vec{v}_1 = \begin{bmatrix}2\\-1\end{bmatrix}}$

$\boxed{\lambda_2 = -5}$ $\begin{cases} (1+5)x + 2y = 0 \\ -3x + (-6+5)y = 0 \end{cases} \Rightarrow \begin{cases} 6x + 2y = 0 \\ -3x - y = 0 \end{cases} \Rightarrow \begin{cases} y = -3x \\ y = -3x \end{cases} \Rightarrow \boxed{\vec{v}_2 = \begin{bmatrix}1\\-3\end{bmatrix}}$

Gen Sol: $\vec{y}(t) = C_1\vec{v}_1 e^{\lambda_1 t} + C_2\vec{v}_2 e^{\lambda_2 t}$

$\Rightarrow \vec{y}(t) = C_1\begin{bmatrix}2\\-1\end{bmatrix} + C_2\begin{bmatrix}1\\-3\end{bmatrix}e^{-5t}$

Unique Sol: $\vec{y}(0) = C_1\begin{bmatrix}2\\-1\end{bmatrix} + C_2\begin{bmatrix}1\\-3\end{bmatrix}$

$\Rightarrow \begin{bmatrix}7\\2\end{bmatrix} = \begin{bmatrix}2C_1 + C_2\\-C_1 - 3C_2\end{bmatrix}$

$\Rightarrow \begin{cases} 2C_1 + C_2 = 7 \\ -2C_1 - 6C_2 = 4 \end{cases} \Rightarrow \begin{cases} -5C_2 = 11 \\ C_1 = -2 - 3C_2 \end{cases} \Rightarrow \begin{cases} C_2 = \frac{11}{-5} \\ C_1 = \frac{23}{5} \end{cases}$

$\Rightarrow$ $\boxed{\begin{aligned} \vec{y}(t) &= \tfrac{23}{5}\begin{bmatrix}2\\-1\end{bmatrix} - \tfrac{11}{5}\begin{bmatrix}1\\-3\end{bmatrix}e^{-5t} \\ x(t) &= \tfrac{46}{5} - \tfrac{11}{5}e^{-5t} \\ y(t) &= -\tfrac{23}{5} + \tfrac{33}{5}e^{-5t} \end{aligned}}$

As $t \to \infty$,

$x(t) \to \frac{46}{5}$

$y(t) \to -\frac{23}{5}$