§3.2 (PART 1): EIGENVALUES AND EIGENVECTORS

1.] A linear system and its vector field are presented below:

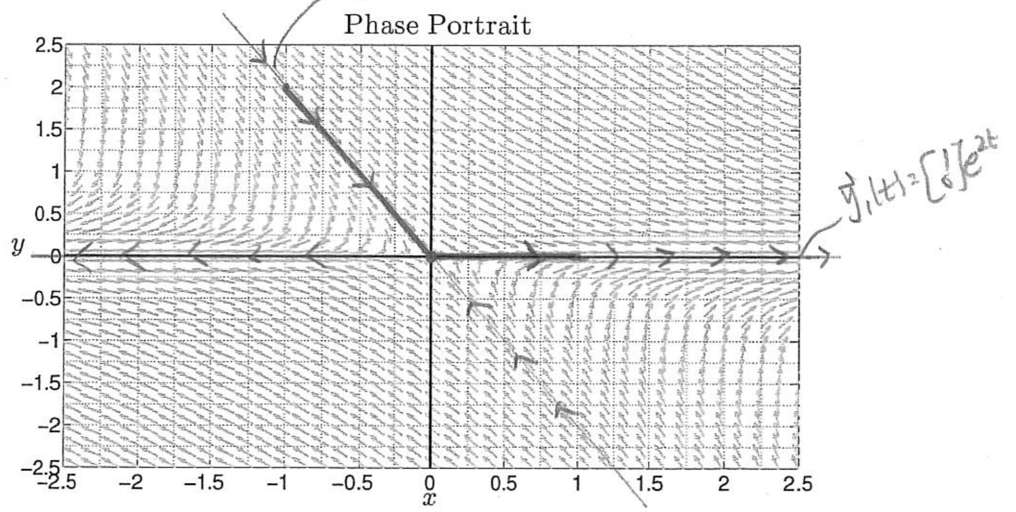
Linear system:

$$\frac{dy}{dt} = \begin{bmatrix} 2 & 3 \\ 0 & -4 \end{bmatrix} y$$

Solutions from previous worksheet:

$$y_1(t) = \begin{bmatrix} e^{2t} \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{2t}$$

$$y_2(t) = \begin{bmatrix} -e^{-4t} \\ 2e^{-4t} \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} e^{-4t}$$



Find the eigenvalues and eigenvectors of the matrix above and use them to construct the general solution to the system.

Eigenvalues: $\det(A-\lambda I) = \det\left(\begin{bmatrix} 2-\lambda & 3 \\ 0 & -4-\lambda \end{bmatrix}\right) = (2-\lambda)(-4-\lambda) - 3(0) = 0$

$\Rightarrow -(2-\lambda)(\lambda+4) = 0$

$\Rightarrow (\lambda-2)(\lambda+4) = 0 \Rightarrow \lambda_1 = 2, \lambda_2 = -4$

Eigenvectors: $\boxed{\lambda_1 = 2}$ $\begin{bmatrix} 2-\lambda_1 & 3 \\ 0 & -4-\lambda_1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{cases} (2-\lambda_1)x + 3y = 0 \\ 0x + (-4-\lambda_1)y = 0 \end{cases} \Rightarrow \begin{cases} 3y = 0 \\ -6y = 0 \end{cases} \Rightarrow \begin{cases} x = \text{free} \\ y = 0 \end{cases}$

$\boxed{\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}}$

$\boxed{\lambda_2 = -4}$ $\begin{bmatrix} 2-\lambda_2 & 3 \\ 0 & -4-\lambda_2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{cases} (2-\lambda_2)x + 3y = 0 \\ 0x + (-4-\lambda_2)y = 0 \end{cases} \Rightarrow \begin{cases} 6x + 3y = 0 \\ 0x + 0y = 0 \end{cases} \Rightarrow \begin{cases} y = -2x \\ 0 = 0 \end{cases}$

$\boxed{\vec{v}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}}$

General Solution: $\vec{y}(t) = c_1\vec{v}_1 e^{\lambda_1 t} + c_2\vec{v}_2 e^{\lambda_2 t} \Rightarrow \boxed{\vec{y}(t) = c_1\begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{2t} + c_2\begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-4t}}$

2.] Find the general solution to the harmonic oscillator:

$$\frac{d^2y}{dt^2} + 7\frac{dy}{dt} + 10y = 0.$$

Convert to a system: $\frac{dy}{dt} = v$ then $\frac{dv}{dt} + 7v + 10y = 0$. We have

$$\begin{aligned} \frac{dy}{dt} &= v \\ \frac{dv}{dt} &= -10y - 7v \end{aligned} \Rightarrow \frac{d\vec{y}}{dt} = \underbrace{\begin{bmatrix} 0 & 1 \\ -10 & -7 \end{bmatrix}}_{A} \vec{y}, \text{ where } \vec{y} = \begin{bmatrix} y(t) \\ v(t) \end{bmatrix}$$

Eigenvalues: $\lambda^2 - tr(A)\lambda + det(A) = 0 \Rightarrow \lambda^2 - (-7)\lambda + 10 = 0$

$\Rightarrow \lambda^2 + 7\lambda + 10 = 0$

$\Rightarrow (\lambda+5)(\lambda+2) = 0 \Rightarrow \boxed{\lambda_1 = -2, \lambda_2 = -5}$

Eigenvectors: $\boxed{\lambda_1 = -2}$ $\begin{bmatrix} 0-\lambda_1 & 1 \\ -10 & -7-\lambda_1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{cases} 2x + y = 0 \\ -10x - 5y = 0 \end{cases} \Rightarrow \begin{cases} y = -2x \\ y = -2x \end{cases}$

$$\boxed{\vec{v}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}}$$

$\boxed{\lambda_2 = -5}$: $\begin{bmatrix} 0-\lambda_2 & 1 \\ -10 & -7-\lambda_2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{cases} 5x + y = 0 \\ -10x - 2y = 0 \end{cases} \Rightarrow \begin{cases} y = -5x \\ y = -5x \end{cases}$

$$\boxed{\vec{v}_2 = \begin{bmatrix} 1 \\ -5 \end{bmatrix}}$$

Gen Sol: $\vec{y}_1(t) = \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-2t}$ $\vec{y}_2 = \begin{bmatrix} 1 \\ -5 \end{bmatrix} e^{-5t}$

$$\boxed{\vec{y}(t) = C_1 \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-2t} + C_2 \begin{bmatrix} 1 \\ -5 \end{bmatrix} e^{-5t}}$$

Note: $y(t) = C_1 e^{-2t} + C_2 e^{-5t}$ we found this using Guess & Test!