§3.2 (PART 2): EIGENVALUES, EIGENVALUES, AND IVPS

1.] Verify that $\lambda_1 = 3$ and $v_1 = \begin{bmatrix} 6 \\ -2 \end{bmatrix}$, and $\lambda_2 = 1$ and $v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ make up two eigenvalue/eigenvector pairs for the matrix $A = \begin{bmatrix} 4 & 3 \\ -1 & 0 \end{bmatrix}$.

$\underline{\lambda_1, \vec{v}_1}$: $A\vec{v}_1 = \begin{bmatrix} 4 & 3 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 6 \\ -2 \end{bmatrix} = \begin{bmatrix} 24-6 \\ -6+0 \end{bmatrix} = \begin{bmatrix} 18 \\ -6 \end{bmatrix}$

$\lambda_1\vec{v}_1 = 3\begin{bmatrix} 6 \\ -2 \end{bmatrix} = \begin{bmatrix} 18 \\ -6 \end{bmatrix}$ — ✓

$\underline{\lambda_2, \vec{v}_2}$: $A\vec{v}_2 = \begin{bmatrix} 4 & 3 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 4-3 \\ -1+0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\lambda_2\vec{v}_2 = 1\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ — ✓

2.] Compute the eigenvalues and eigenvectors of the matrix $A = \begin{bmatrix} -3 & 1 \\ -1 & 1 \end{bmatrix}$.

$\underline{\text{E-vals}}$: $tr(A) = -3+1 = -2$ $\quad det(A) = -3-(-1) = -2$

$\lambda^2 - tr(A)\lambda + det(A) = \lambda^2 + 2\lambda - 2 = 0$

$\Rightarrow \lambda = \frac{-2}{2} \pm \frac{\sqrt{4-(4)(1)(-2)}}{2} \Rightarrow \lambda = -1 \pm \frac{\sqrt{12}}{3}$

$\Rightarrow \boxed{\lambda_1 = -1+\sqrt{3},\ \lambda_2 = -1-\sqrt{3}}$

$\underline{\text{E-vectors}}$:

$\boxed{\lambda_1 = -1+\sqrt{3}}$ $\begin{cases} (-3-\lambda_1)x + y = 0 \\ -1x + (1-\lambda_1)y = 0 \end{cases} \Rightarrow \begin{cases} (-2-\sqrt{3})x + y = 0 \\ -x + (2-\sqrt{3})y = 0 \end{cases} \Rightarrow \begin{cases} y = (2+\sqrt{3})x \\ y = \frac{1}{2-\sqrt{3}}x \end{cases}$

$\boxed{\vec{v}_1 = \begin{bmatrix} 1 \\ 2+\sqrt{3} \end{bmatrix}}$ $\Rightarrow \begin{cases} y = (2+\sqrt{3})x \\ y = \frac{1}{2-\sqrt{3}}\left(\frac{2+\sqrt{3}}{2+\sqrt{3}}\right)x \end{cases} \Rightarrow \begin{cases} y = (2+\sqrt{3})x \\ y = (2+\sqrt{3})x \end{cases}$

$\boxed{\lambda_2 = -1-\sqrt{3}}$ $\begin{cases} (-3-\lambda_2)x + y = 0 \\ -x + (1-\lambda_2)y = 0 \end{cases} \Rightarrow \begin{cases} (-2+\sqrt{3})x + y = 0 \\ -x + (2+\sqrt{3})y = 0 \end{cases} \Rightarrow \begin{cases} y = (2-\sqrt{3})x \\ y = \frac{1}{2+\sqrt{3}}x \end{cases}$

$\boxed{\vec{v}_2 = \begin{bmatrix} 1 \\ 2-\sqrt{3} \end{bmatrix}}$ $\Rightarrow \begin{cases} y = (2-\sqrt{3})x \\ y = \frac{1}{2+\sqrt{3}}\left(\frac{2-\sqrt{3}}{2-\sqrt{3}}\right)x \end{cases} \Rightarrow \begin{cases} y = (2-\sqrt{3})x \\ y = (2-\sqrt{3})x \end{cases}$

3.] Solve the following initial-value problem

$$\frac{dy}{dt} = \begin{bmatrix} 3 & 0 \\ 1 & -2 \end{bmatrix} y, \qquad y(0) = \begin{bmatrix} 2 \\ 2 \end{bmatrix}.$$

E-values: $A = \begin{bmatrix} 3 & 0 \\ 1 & -2 \end{bmatrix}$ $\lambda^2 - tr(A)\lambda + det(A) = 0$

$tr(A) = 3-2 = 1$ $\Rightarrow \lambda^2 - \lambda - 6 = 0$

$det(A) = -6 - 0 = -6$ $(\lambda-3)(\lambda+2) = 0$ $\Rightarrow$ $\boxed{\lambda_1 = -2, \lambda_2 = 3}$.

E-vectors: $\boxed{\lambda_1 = -2}$ $\begin{cases} (3-(-2))x + 0y = 0 \\ x + (-2-(-2))y = 0 \end{cases} \Rightarrow \begin{cases} 5x = 0 \\ x + 0y = 0 \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = \text{free} \end{cases}$

$\boxed{\vec{v}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}}$

$\boxed{\lambda_2 = 3}$ $\begin{cases} (3-3)x + 0y = 0 \\ x + (-2-3)y = 0 \end{cases} \Rightarrow \begin{cases} 0 = 0 \\ x - 5y = 0 \end{cases} \Rightarrow \begin{cases} 0 = 0 \\ y = \frac{1}{5}x \end{cases}$

$\boxed{\vec{v}_2 = \begin{bmatrix} 5 \\ 1 \end{bmatrix}}$

Gen Sol: straight-Line Solutions: $\vec{y}_1(t) = \vec{v}_1 e^{\lambda_1 t}$ $\vec{y}_2(t) = \vec{v}_2 e^{\lambda_2 t}$

$\vec{y}_1(t) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-2t}$ $\vec{y}_2(t) = \begin{bmatrix} 5 \\ 1 \end{bmatrix} e^{3t}$

$$\boxed{y(t) = C_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-2t} + C_2 \begin{bmatrix} 5 \\ 1 \end{bmatrix} e^{3t}}$$

$\Rightarrow$ $x(t) = 5C_2 e^{3t}$

$y(t) = C_1 e^{-2t} + C_2 e^{3t}$