§3.1: Linear Systems

1.] Let $A = \begin{bmatrix} 5 & 2 \\ -1 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 2 & \pi \\ e^5 & 6 \end{bmatrix}$, $\boldsymbol{x} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$, and $\boldsymbol{y} = \begin{bmatrix} -1 \\ 5 \end{bmatrix}$. Compute $A\boldsymbol{x}$, AB, $A(\boldsymbol{x}+\boldsymbol{y})$, and $\boldsymbol{xy}$.

$$A\vec{x} = \begin{bmatrix} 5 & 2 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 5(3)+2(4) \\ -1(3)+3(4) \end{bmatrix} = \begin{bmatrix} 15+8 \\ -3+12 \end{bmatrix} = \begin{bmatrix} 23 \\ 9 \end{bmatrix}$$

$$AB = \begin{bmatrix} 5 & 2 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 2 & \pi \\ e^5 & 6 \end{bmatrix} = \begin{bmatrix} (10+2e^5) & (5\pi+(12)) \\ (-2+e^5) & (-\pi+18) \end{bmatrix}$$

$$A(\vec{x}+\vec{y}) = \begin{bmatrix} 5 & 2 \\ -1 & 3 \end{bmatrix}\left(\begin{bmatrix} 3 \\ 4 \end{bmatrix}+\begin{bmatrix} -1 \\ 5 \end{bmatrix}\right) = \begin{bmatrix} 5 & 2 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 2 \\ 9 \end{bmatrix} = \begin{bmatrix} 10+18 \\ -2+27 \end{bmatrix} = \begin{bmatrix} 28 \\ 25 \end{bmatrix}$$

$\vec{x}\vec{y} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}\begin{bmatrix} -1 \\ 5 \end{bmatrix} =$ error (inner dimensions don't agree)

2.] Consider the following system:

$$\begin{aligned} \frac{dx}{dt} &= 5z \\ \frac{dy}{dt} &= -5x + y \\ \frac{dz}{dt} &= -y + 6z \end{aligned}$$

Define $\boldsymbol{y}$ and A so that this system is represented as $\dfrac{d\boldsymbol{y}}{dt} = A\boldsymbol{y}$.

$$\frac{dx}{dt} = 0x + 0y + 5z$$

$$\frac{dy}{dt} = -5x + y + 0z$$

$$\frac{dz}{dt} = 0x - y + 6z$$

Let $\vec{y}(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}$ and $A = \begin{bmatrix} 0 & 0 & 5 \\ -5 & 1 & 0 \\ 0 & -1 & 6 \end{bmatrix}$

Then $\dfrac{d\vec{y}}{dt} = A\vec{y}$.

3.] Consider the system

$$\frac{dy}{dt} = \underbrace{\begin{bmatrix} 2 & 3 \\ 0 & -4 \end{bmatrix}}_{A} y.$$

Show that both of the solutions below satisfy this equation. Then show that $y(t) = -y_1(t) + 3y_2(t)$ also satisfies this equation.

$$y_1(t) = \begin{bmatrix} e^{2t} \\ 0 \end{bmatrix}, \qquad y_2(t) = \begin{bmatrix} -e^{-4t} \\ 2e^{-4t} \end{bmatrix}.$$

$\underline{\text{LHS}}$: $\frac{d\vec{y}_1}{dt} = \frac{d}{dt}\begin{bmatrix} e^{2t} \\ 0 \end{bmatrix} = \begin{bmatrix} 2e^{2t} \\ 0 \end{bmatrix}$ ✓

$\underline{\text{RHS}}$: $A\vec{y}_1 = \begin{bmatrix} 2 & 3 \\ 0 & -4 \end{bmatrix}\begin{bmatrix} e^{2t} \\ 0 \end{bmatrix} = \begin{bmatrix} 2e^{2t} + 3(0) \\ 0e^{2t} + (-4)(0) \end{bmatrix} = \begin{bmatrix} 2e^{2t} \\ 0 \end{bmatrix}$

$\underline{\text{LHS}}$: $\frac{d\vec{y}_2}{dt} = \frac{d}{dt}\begin{bmatrix} -e^{-4t} \\ 2e^{-4t} \end{bmatrix} = \begin{bmatrix} 4e^{-4t} \\ -8e^{-4t} \end{bmatrix}$ ✓

$\underline{\text{RHS}}$: $A\vec{y}_2 = \begin{bmatrix} 2 & 3 \\ 0 & -4 \end{bmatrix}\begin{bmatrix} -e^{-4t} \\ 2e^{-4t} \end{bmatrix} = \begin{bmatrix} -2e^{-4t} + 6e^{-4t} \\ -0e^{-4t} - 8e^{-4t} \end{bmatrix} = \begin{bmatrix} 4e^{-4t} \\ -8e^{-4t} \end{bmatrix}$

$\underline{\text{LHS}}$: $\frac{d\vec{y}}{dt} = \frac{d}{dt}\left(-\vec{y}_1(t) + 3\vec{y}_2(t)\right) = -\frac{d\vec{y}_1}{dt} + \frac{3d\vec{y}_2}{dt}$ From above!

$= -(A\vec{y}_1) + 3(A\vec{y}_2)$

$= A(-\vec{y}_1 + 3\vec{y}_2)$

$= A\vec{y}$: $\underline{\text{RHS}}$

In fact, for any k_1, k_2, we have $\vec{y}(t) = k_1\vec{y}_1(t) + k_2\vec{y}_2(t)$. Then

$$\frac{d\vec{y}}{dt} = \frac{d}{dt}(k_1\vec{y}_1 + k_2\vec{y}_2) = k_1\frac{d\vec{y}_1}{dt} + k_2\frac{d\vec{y}_2}{dt} = k_1A\vec{y}_1 + k_2A\vec{y}_2$$

$$= A(k_1\vec{y}_1 + k_2\vec{y}_2) = A\vec{y}$$ ✓