§3.4: Solutions and Stability - Complex Eigenvalues

Find the general solution to the following linear system of equations:

$$\frac{dy}{dt} = \begin{bmatrix} -2 & -3 \\ 3 & -2 \end{bmatrix} y$$

E-values: $\lambda^2 - tr(A)\lambda + det(A) = 0$

$\lambda^2 + 4\lambda + 13 = 0$

$\lambda = \frac{-4}{2} \pm \frac{\sqrt{(4)^2 - 4(13)}}{2}$

$\Rightarrow \lambda = -2 \pm \frac{\sqrt{-36}}{2}$

$\Rightarrow \lambda = -2 \pm 3i$

$\Rightarrow \boxed{\lambda_1 = -2+3i, \ \lambda_2 = -2-3i}$

Spiral Sink

E-vector: $\boxed{\lambda_1 = -2+3i}$

$\begin{cases} (-2-(-2+3i))x - 3y = 0 \\ 3x + (-2-(-2+3i))y = 0 \end{cases} \Rightarrow \begin{cases} -3ix - 3y = 0 \\ 3x - 3iy = 0 \end{cases} \Rightarrow \begin{cases} y = -ix \\ y = \frac{1}{i}x = -ix \end{cases}$

$\boxed{\vec{v}_1 = \begin{bmatrix} 1 \\ -i \end{bmatrix}}$

note: You don't have to find $\vec{v}_2$.

Construct $\vec{y}_1(t)$: $\vec{y}_1(t) = \vec{v}_1 e^{\lambda_1 t}$

$\Rightarrow \vec{y}_1(t) = \begin{bmatrix} 1 \\ -i \end{bmatrix} e^{(-2+3i)t}$

$\Rightarrow \vec{y}_1(t) = \begin{bmatrix} 1 \\ -i \end{bmatrix} e^{-2t} e^{i3t}$

$\Rightarrow \vec{y}_1(t) = \begin{bmatrix} 1 \\ -i \end{bmatrix} e^{-2t}(\cos(3t) + i\sin(3t))$

$\Rightarrow \vec{y}_1(t) = \begin{bmatrix} e^{-2t}\cos(3t) + ie^{-2t}\sin(3t) \\ -ie^{-2t}\cos(3t) + e^{-2t}\sin(3t) \end{bmatrix}$

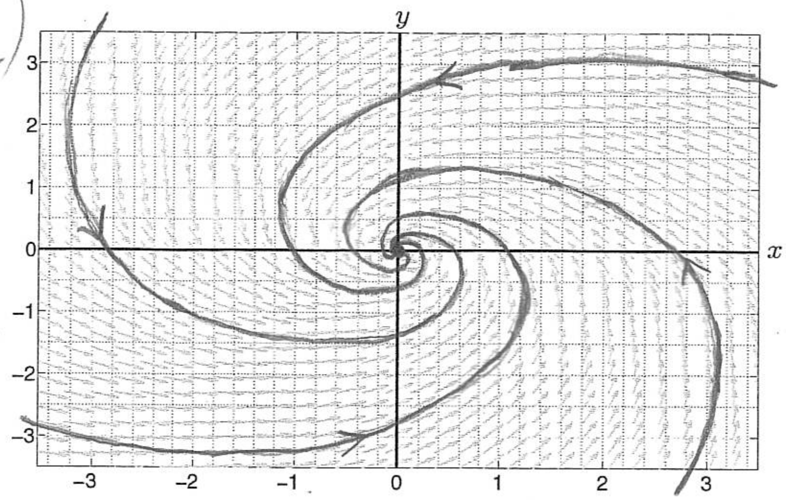
$\Rightarrow \vec{y}_1(t) = \underbrace{\begin{bmatrix} \cos(3t) \\ \sin(3t) \end{bmatrix} e^{-2t}}_{\vec{y}_R(t)} + i \underbrace{\begin{bmatrix} \sin(3t) \\ -\cos(3t) \end{bmatrix} e^{-2t}}_{\vec{y}_I(t)}$

General Solution: $\vec{y}(t) = C_1 \vec{y}_R(t) + C_2 \vec{y}_I(t)$

$\Rightarrow \vec{y}(t) = C_1 \begin{bmatrix} \cos(3t) \\ \sin(3t) \end{bmatrix} e^{-2t} + C_2 \begin{bmatrix} \sin(3t) \\ -\cos(3t) \end{bmatrix} e^{-2t}$

$\Rightarrow x(t) = e^{-2t}(C_1\cos(3t) + C_2\sin(2t)) \quad y(t) = e^{-2t}(C_1\sin(3t) - C_2\cos(3t))$

Spiral Sink



Consider the system:

$$\frac{d\mathbf{y}}{dt} = \begin{bmatrix} -2 & -3 \\ 3 & -2 \end{bmatrix} \mathbf{y}$$

Eigenvalues: $\lambda_1 = -2+3i$, $\lambda_2 = -2-3i$

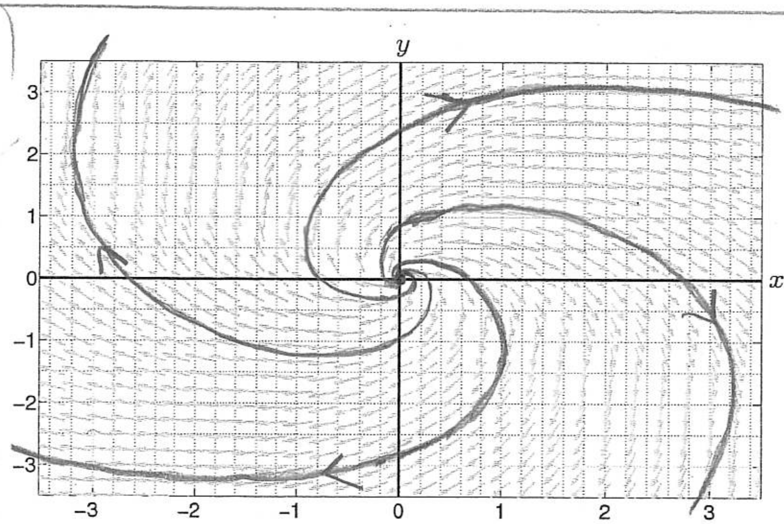
Eigenvectors: $\vec{v}_1 = \begin{bmatrix} 1 \\ -i \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ i \end{bmatrix}$

Gen. Sol. : $\vec{y}(t) = C_1 \begin{bmatrix} \cos(3t) \\ \sin(3t) \end{bmatrix} e^{-2t} + C_2 \begin{bmatrix} \sin(3t) \\ -\cos(3t) \end{bmatrix} e^{-2t}$

$x(t) = e^{-2t}(C_1\cos(3t) + C_2\sin(3t))$

$y(t) = e^{-2t}(C_1\sin(3t) - C_2\cos(3t))$

Spiral Source



Consider the system:

$$\frac{d\mathbf{y}}{dt} = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix} \mathbf{y}$$

Eigenvalues: $\lambda_1 = 2+3i$, $\lambda_2 = 2-3i$

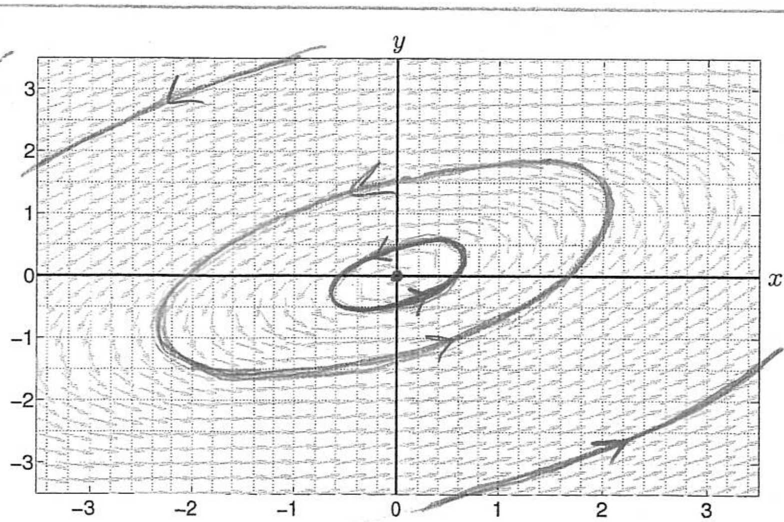
Eigenvectors: $\vec{v}_1 = \begin{bmatrix} 1 \\ i \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ -i \end{bmatrix}$

Gen. Sol. : $\vec{y}(t) = C_1 \begin{bmatrix} \cos(3t) \\ -\sin(3t) \end{bmatrix} e^{2t} + C_2 \begin{bmatrix} \sin(3t) \\ \cos(3t) \end{bmatrix} e^{2t}$

$x(t) = e^{2t}(C_1\cos(3t) + C_2\sin(3t))$

$y(t) = e^{2t}(-C_1\sin(3t) + C_2\cos(3t))$

Center



Consider the system:

$$\frac{d\mathbf{y}}{dt} = \begin{bmatrix} 3 & -9 \\ 4 & -3 \end{bmatrix} \mathbf{y}$$

Eigenvalues: $\lambda_1 = 3\sqrt{3}i$, $\lambda_2 = -3\sqrt{3}i$

Eigenvectors: $\vec{v}_1 = \begin{bmatrix} 1 \\ \frac{1}{3} - \frac{\sqrt{3}}{3}i \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ \frac{1}{3} + \frac{\sqrt{3}}{3}i \end{bmatrix}$

Gen. Sol. : See next page

$x(t) =$ See next page

$y(t) =$ See next page.

Back of Section 3.4 Worksheet:

$$\frac{d\vec{y}}{dt} = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix} \vec{y}$$

$tr(A) = 4$

$det(A) = 13$

E-values: $\lambda^2 - 4\lambda + 13 = 0$

$$\lambda = \frac{4}{2} \pm \frac{\sqrt{(-4)^2 - 4(13)}}{2}$$

$$\lambda = \frac{4}{2} \pm \frac{\sqrt{-36}}{2} \Rightarrow \boxed{\lambda_1 = 2+3i, \ \lambda_2 = 2-3i}$$

Spiral Source

E-vectors: $\boxed{\lambda_1 = 2+3i}$

$$\begin{cases} (2-(2+3i))x + 3y = 0 \\ -3x + (2-(2+3i))y = 0 \end{cases} \Rightarrow \begin{cases} -3ix + 3y = 0 \\ -3x - 3iy = 0 \end{cases} \Rightarrow \begin{cases} y = ix \\ y = -\frac{1}{i}x = ix \end{cases}$$

$$\boxed{\vec{v}_1 = \begin{bmatrix} 1 \\ i \end{bmatrix}}$$

$\boxed{\lambda_2 = 2-3i}$

$$\begin{cases} (2-(2-3i))x + 3y = 0 \\ -3x + (2-(2-3i))y = 0 \end{cases} \Rightarrow \begin{cases} 3ix + 3y = 0 \\ -3x + 3iy = 0 \end{cases} \Rightarrow \begin{cases} y = -ix \\ y = \frac{1}{i}x = -ix \end{cases}$$

$$\boxed{\vec{v}_2 = \begin{bmatrix} 1 \\ -i \end{bmatrix}}$$

Unnecessary (Don't need)

Construct $\vec{y}_1(t)$: $\vec{y}_1(t) = \vec{v}_1 e^{\lambda_1 t}$

$$\Rightarrow \vec{y}_1(t) = \begin{bmatrix} 1 \\ i \end{bmatrix} e^{(2+3i)t}$$

$$\Rightarrow \vec{y}_1(t) = \begin{bmatrix} 1 \\ i \end{bmatrix} e^{2t}(\cos(3t) + i\sin(3t))$$

$$\Rightarrow \vec{y}_1(t) = \begin{bmatrix} e^{2t}\cos(3t) + i e^{2t}\sin(3t) \\ ie^{2t}\cos(3t) - e^{2t}\sin(3t) \end{bmatrix}$$

$$\Rightarrow \vec{y}_1(t) = \underbrace{\begin{bmatrix} \cos(3t) \\ -\sin(3t) \end{bmatrix} e^{2t}}_{\vec{y}_R(t)} + i\underbrace{\begin{bmatrix} \sin(3t) \\ \cos(3t) \end{bmatrix} e^{2t}}_{\vec{y}_I(t)}$$

Gen. Sol: $\vec{y}(t) = C_1 \begin{bmatrix} \cos(3t) \\ -\sin(3t) \end{bmatrix} e^{2t} + C_2 \begin{bmatrix} \sin(3t) \\ \cos(3t) \end{bmatrix} e^{2t}$

$$x(t) = e^{2t}(C_1\cos(3t) + C_2\sin(3t)) \qquad y(t) = e^{2t}(-C_1\sin(3t) + C_2\cos(3t))$$

$$\frac{d\vec{y}}{dt} = \begin{bmatrix} 3 & -9 \\ 4 & -3 \end{bmatrix}\vec{y}$$

$tr(A) = 0$

$det(A) = 27$

E-values: $\lambda^2 + 27 = 0$

$\lambda^2 = -27$

$\lambda = \pm\sqrt{-27} \Rightarrow \boxed{\lambda_1 = 3\sqrt{3}i, \ \lambda_2 = -3\sqrt{3}i}$

Center

E-vectors: $\boxed{\lambda_1 = 3\sqrt{3}i}$ $\begin{cases} (3-3\sqrt{3}i)x - 9y = 0 \\ 4x + (-3-3\sqrt{3}i)y = 0 \end{cases} \Rightarrow \begin{cases} y = \left(\frac{3-3\sqrt{3}i}{9}\right)x \\ y = \frac{4}{3+3\sqrt{3}i}x \end{cases} \Rightarrow \begin{cases} y = \left(\frac{1}{3} - \frac{\sqrt{3}}{3}i\right)x \\ y = \left(\frac{1}{3} - \frac{\sqrt{3}}{3}i\right)x \end{cases}$

$$\boxed{\vec{v}_1 = \begin{bmatrix} 1 \\ \frac{1}{3} - \frac{\sqrt{3}}{3}i \end{bmatrix}}$$

unnecessary!

$\boxed{\lambda_2 = -3\sqrt{3}i}$ $\begin{cases} (3+3\sqrt{3}i)x - 9y = 0 \\ 4x + (-3+3\sqrt{3}i)y = 0 \end{cases} \Rightarrow \begin{cases} y = \frac{3+3\sqrt{3}i}{9}x \\ y = \frac{4}{3-3\sqrt{3}i}x \end{cases} \Rightarrow \begin{cases} y = \left(\frac{1}{3} + \frac{\sqrt{3}}{3}i\right)x \\ y = \left(\frac{1}{3} + \frac{\sqrt{3}}{3}i\right)x \end{cases}$

$$\boxed{\vec{v}_2 = \begin{bmatrix} 1 \\ \frac{1}{3} + \frac{\sqrt{3}}{3}i \end{bmatrix}}$$

Construct $\vec{y}_1(t)$: $\vec{y}_1(t) = \vec{v}_1 e^{\lambda_1 t}$

$$\Rightarrow \vec{y}_1(t) = \begin{bmatrix} 1 \\ \frac{1}{3} - \frac{\sqrt{3}}{3}i \end{bmatrix} e^{3\sqrt{3}it}$$

$$\Rightarrow \vec{y}_1(t) = \begin{bmatrix} 1 \\ \frac{1}{3} - \frac{\sqrt{3}}{3}i \end{bmatrix} (\cos(3\sqrt{3}t) + i\sin(3\sqrt{3}t))$$

$$\Rightarrow \vec{y}_1(t) = \begin{bmatrix} \cos(3\sqrt{3}t) + i\sin(3\sqrt{3}t) \\ \frac{1}{3}\cos(3\sqrt{3}t) - i\frac{\sqrt{3}}{3}\cos(3\sqrt{3}t) + i\frac{1}{3}\sin(3\sqrt{3}t) + \frac{\sqrt{3}}{3}\sin(3\sqrt{3}t) \end{bmatrix}$$

$$\Rightarrow \vec{y}_1(t) = \underbrace{\begin{bmatrix} \cos(3\sqrt{3}t) \\ \frac{1}{3}\cos(3\sqrt{3}t) + \frac{\sqrt{3}}{3}\sin(3\sqrt{3}t) \end{bmatrix}}_{\vec{y}_R(t)} + i\underbrace{\begin{bmatrix} \sin(3\sqrt{3}t) \\ -\frac{\sqrt{3}}{3}\cos(3\sqrt{3}t) + \frac{1}{3}\sin(3\sqrt{3}t) \end{bmatrix}}_{\vec{y}_I(t)}$$

(3)

<u>Gen Sol</u>: $\vec{y}(t) = C_1 \begin{bmatrix} \cos(3\sqrt{3}t) \\ \frac{1}{3}\cos(3\sqrt{3}t) + \frac{\sqrt{3}}{3}\sin(3\sqrt{3}t) \end{bmatrix} + C_2 \begin{bmatrix} \sin(3\sqrt{3}t) \\ -\frac{\sqrt{3}}{3}\cos(3\sqrt{3}t) + \frac{1}{3}\sin(3\sqrt{3}t) \end{bmatrix}$

$$x(t) = C_1 \cos(3\sqrt{3}t) + C_2 \sin(3\sqrt{3}t)$$

$$y(t) = C_1\left(\frac{1}{3}\cos(3\sqrt{3}t) + \frac{\sqrt{3}}{3}\sin(3\sqrt{3}t)\right) + C_2\left(-\frac{\sqrt{3}}{3}\cos(3\sqrt{3}t) + \frac{1}{3}\sin(3\sqrt{3}t)\right)$$