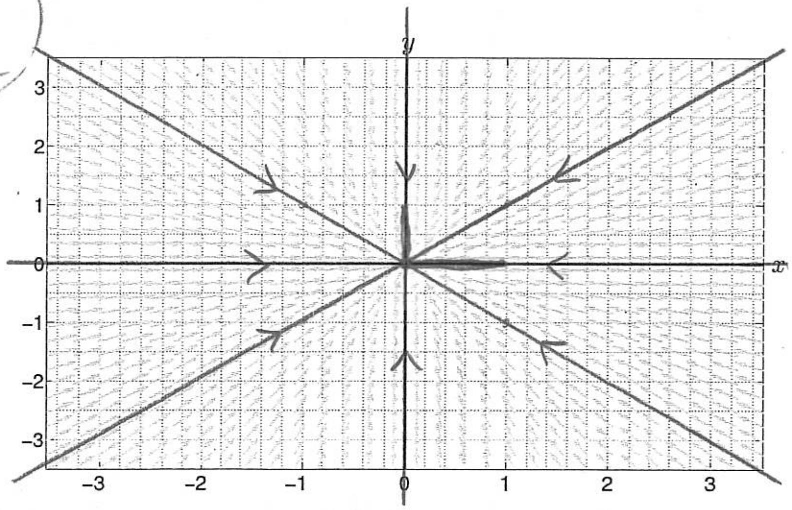
§3.5 (Part 1): Stability - Repeated Eigenvalues

Star Sink



Consider the system:

$$\frac{d\mathbf{y}}{dt} = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} \mathbf{y}$$

Eigenvalues: $\lambda_1 = -2, \ \lambda_2 = -2$

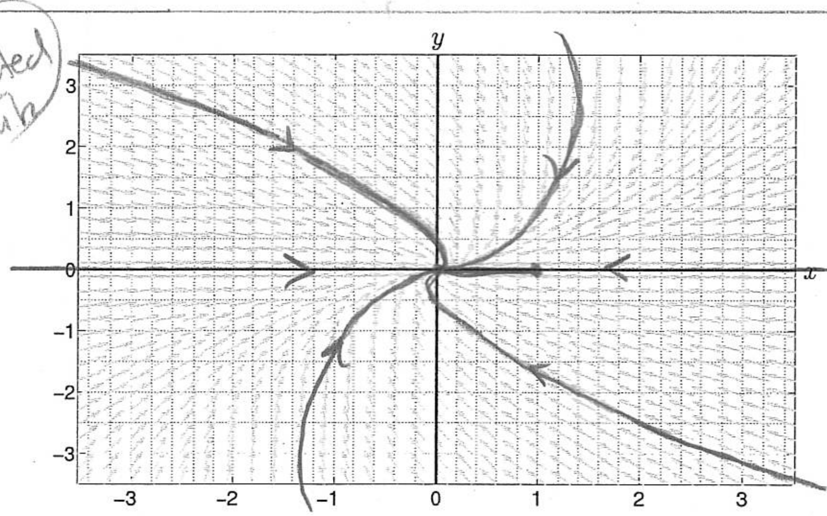
Eigenvectors: $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Gen. Sol. : $\vec{y}(t) = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-2t}$

$x(t) = c_1 e^{-2t}$

$y(t) = c_2 e^{-2t}$

Repeated Sink



Consider the system:

$$\frac{d\mathbf{y}}{dt} = \begin{bmatrix} -2 & 1 \\ 0 & -2 \end{bmatrix} \mathbf{y}$$

Eigenvalues: $\lambda_1 = -2, \ \lambda_2 = -2$

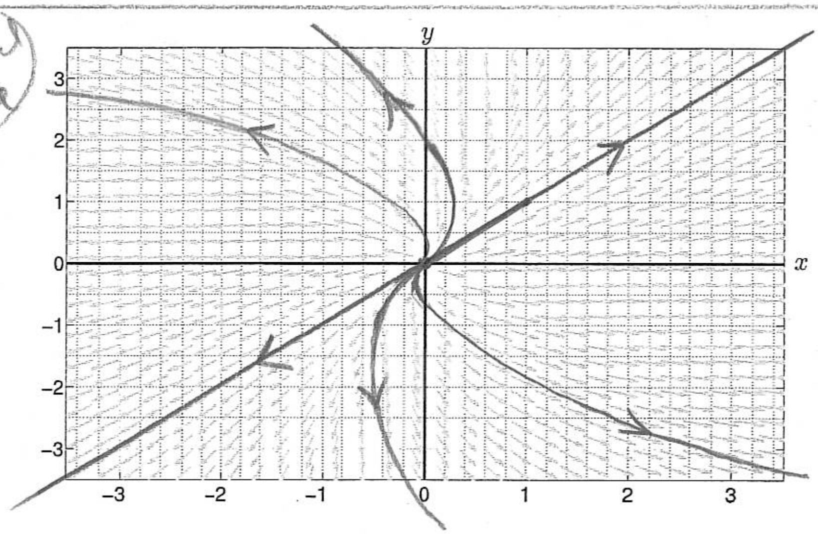
Eigenvectors: $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, Gen Evector: $\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Gen. Sol. : $\vec{y}(t) = \begin{bmatrix} c_1 + c_2 \\ c_2 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} t e^{-2t}$

$x(t) = (c_1 + c_2) e^{-2t} + c_2 t e^{-2t}$

$y(t) = c_2 e^{-2t}$

Repeated Source



Consider the system:

$$\frac{d\mathbf{y}}{dt} = \begin{bmatrix} 5 & -1 \\ 1 & 3 \end{bmatrix} \mathbf{y}$$

Eigenvalues: $\lambda_1 = 4, \ \lambda_2 = 4$

Eigenvectors: $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ Gen. Evector: $\vec{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Gen. Sol. : $\vec{y}(t) = \begin{bmatrix} c_1 + c_2 \\ c_1 \end{bmatrix} e^{4t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} t e^{4t}$

$x(t) = (c_1 + c_2) e^{4t} + c_2 t e^{4t}$

$y(t) = c_1 e^{4t} + c_2 t e^{4t}$

Find the unique solution to the following IVP:

$$\frac{dy}{dt} = \begin{bmatrix} -2 & -1 \\ 1 & -4 \end{bmatrix} \boldsymbol{y}, \qquad \boldsymbol{y}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

E-values: $\lambda^2 - tr(A)\lambda + det(A) = 0$

$\Rightarrow \lambda^2 + 6\lambda + 9 = 0$

$\Rightarrow (\lambda+3)(\lambda+3) = 0$

$\boxed{\lambda_1 = \lambda_2 = -3}$

E-vectors: $\boxed{\lambda_1 = -3}$ $\begin{cases} (-2-(-3))x - y = 0 \\ x + (-4-(-3))y = 0 \end{cases} \Rightarrow \begin{cases} x - y = 0 \\ x - y = 0 \end{cases} \Rightarrow \begin{cases} y = x \\ y = x \end{cases}$

$\boxed{\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$

Defective $\lambda_1 = \lambda_2 = -3$ $\Rightarrow$ $\boxed{\text{Repeated Sink}}$

Gen. E-vector: $\begin{cases} (-2-(-3))x - y = 1 \\ x + (-4-(-3))y = 1 \end{cases} \Rightarrow \begin{cases} x - y = 1 \\ x - y = 1 \end{cases} \Rightarrow \begin{cases} y = x-1 \\ y = x-1 \end{cases}$ $\boxed{\vec{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}}$.

Gen. Sol: $\vec{y}(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-3t} + c_2 \left(t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) e^{-3t}$

$\Rightarrow \vec{y}(t) = \begin{bmatrix} c_1 + c_2 \\ c_1 \end{bmatrix} e^{-3t} + \begin{bmatrix} c_2 \\ c_2 \end{bmatrix} t e^{-3t}$

Unique Sol: $\vec{y}(0) = \begin{bmatrix} c_1 + c_2 \\ c_1 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} c_1 + c_2 \\ c_1 \end{bmatrix}$

$c_1 = 0$

$c_2 = 1$

$\vec{y}(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{-3t} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} t e^{-3t}$

$$\boxed{\begin{aligned} x(t) &= e^{-3t} + te^{-3t} \\ y(t) &= te^{-3t} \end{aligned}}$$

Back of Section 3.5 (Part 1) Worksheet: ①

$$\frac{d\vec{y}}{dt} = \begin{bmatrix} -2 & 1 \\ 0 & -2 \end{bmatrix} \vec{y}$$

$tr(A) = -4$

$det(A) = 4$

E-values: $\lambda^2 + 4\lambda + 4 = 0$

$(\lambda+2)(\lambda+2) = 0$

$\boxed{\lambda_1 = -2, \lambda_2 = -2}$

E-vectors: $\boxed{\lambda_1 = -2}$ $\begin{cases} (-2-(-2))x + y = 0 \\ 0x + (-2-(-2))y = 0 \end{cases} \Rightarrow \begin{cases} 0x + y = 0 \\ 0 = 0 \end{cases} \Rightarrow \begin{cases} y = 0 \\ x = \text{free} \end{cases}$

$\boxed{\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}}$

$\boxed{\begin{matrix}\text{Defective Case} \\ \lambda_1 = \lambda_2 = -2\end{matrix}} \Rightarrow \boxed{\text{Repeated Sink!}}$

Generalized E-vector:

$\begin{cases} (-2-(-2))x + y = 1 \\ 0x + (-2-(-2))y = 0 \end{cases} \Rightarrow \begin{cases} y = 1 \\ 0 = 0 \end{cases} \Rightarrow \begin{cases} y = 1 \\ x = \text{free} \end{cases} \Rightarrow \boxed{\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$

Gen. Sol: $\vec{y}(t) = c_1 \vec{v}_1 e^{\lambda_1 t} + c_2 (t\vec{v}_1 + \vec{v}_2) e^{\lambda_2 t}$

$\Rightarrow \vec{y}(t) = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{-2t} + c_2 \left(t \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) e^{-2t}$

$\Rightarrow \boxed{\vec{y}(t) = \begin{bmatrix} c_1 + c_2 \\ c_2 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} t e^{-2t}}$

$\Rightarrow x(t) = (c_1 + c_2) e^{-2t} + c_2 t e^{-2t}$

$y(t) = c_2 e^{-2t}$

$$\frac{d\vec{y}}{dt} = \begin{bmatrix} 5 & -1 \\ 1 & 3 \end{bmatrix}\vec{y}$$

E-values: $\lambda^2 - 8\lambda + 16 = 0$

$(\lambda-4)(\lambda-4) = 0$

$\boxed{\lambda_1 = 4,\ \lambda_2 = 4}$

$tr(A) = 8$

$det(A) = 16$

E-vectors: $\boxed{\lambda_1 = 4}$ $\begin{cases} (5-4)x - y = 0 \\ x + (3-4)y = 0 \end{cases} \Rightarrow \begin{cases} x - y = 0 \\ x - y = 0 \end{cases} \Rightarrow \begin{cases} y = x \\ y = x \end{cases}$

$\boxed{\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$ Defective Case $\Rightarrow$ $\boxed{\text{Repeated Source}}$

$\lambda_1 = \lambda_2 = 4$

Generalized E-vector: $\begin{cases} (5-4)x - y = 1 \\ x + (3-4)y = 1 \end{cases} \Rightarrow \begin{cases} x - y = 1 \\ x - y = 1 \end{cases} \Rightarrow \begin{cases} y = -1 + x \\ y = -1 + x \end{cases}$ $\boxed{\vec{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}}$

General Solution: $\vec{y}(t) = c_1\vec{v}_1 e^{\lambda_1 t} + c_2(t\vec{v}_1 + \vec{v}_2)e^{\lambda_2 t}$

$\Rightarrow \vec{y}(t) = c_1\begin{bmatrix} 1 \\ 1 \end{bmatrix}e^{4t} + c_2\left(t\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)e^{4t}$

$\Rightarrow \boxed{\vec{y}(t) = \begin{bmatrix} c_1 + c_2 \\ c_1 \end{bmatrix}e^{4t} + c_2\begin{bmatrix} 1 \\ 1 \end{bmatrix}te^{4t}}$

$x(t) = (c_1 + c_2)e^{4t} + c_2 te^{4t}$

$y(t) = c_1 e^{4t} + c_2 te^{4t}$