MT 181-050 - Calculus I
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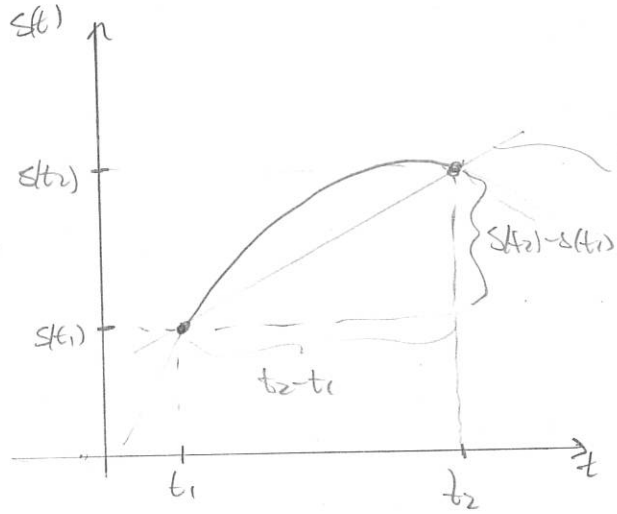
(1)

CHAPTER 3: Derivatives

Section 3.1: Introducing the Derivative

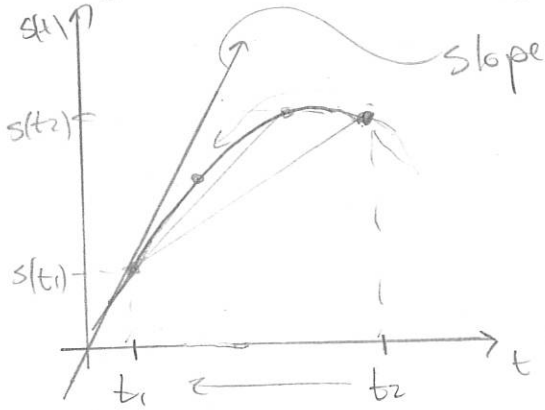
- we return to the age old problem of finding the instantaneous velocity of a moving object. As we've seen before, we can calculate the average velocity over an interval for any position function, $s(t)$:



Slope of line $= \dfrac{s(t_2)-s(t_1)}{t_2-t_1}$

$=$ Avg Velocity over $[t_1, t_2]$

- We understand the average velocity over $[t_1, t_2]$ is exactly equal to the slope of the line that connects both points $(t_1, s(t_1))$ and $(t_2, s(t_2))$.
- We call this line the secant line as it intersects the graph of $s(t)$ at two points.
- Now that we know more about limits, we can more efficiently define the instantaneous velocity of $s(t)$ at a single point:



Slope of Line $= \lim\limits_{t_2 \to t_1} \dfrac{s(t_2)-s(t_1)}{t_2-t_1}$

$=$ Instantaneous velocity at t_1

- We understand that the instantaneous velocity at t_1 is exactly equal to the slope of the tangent line, which only intersects the graph at a single point $(t_1, s(t_1))$.

- The ideas of average and instantaneous velocities follow through to any given function f:

$s(t)$ = position at time t

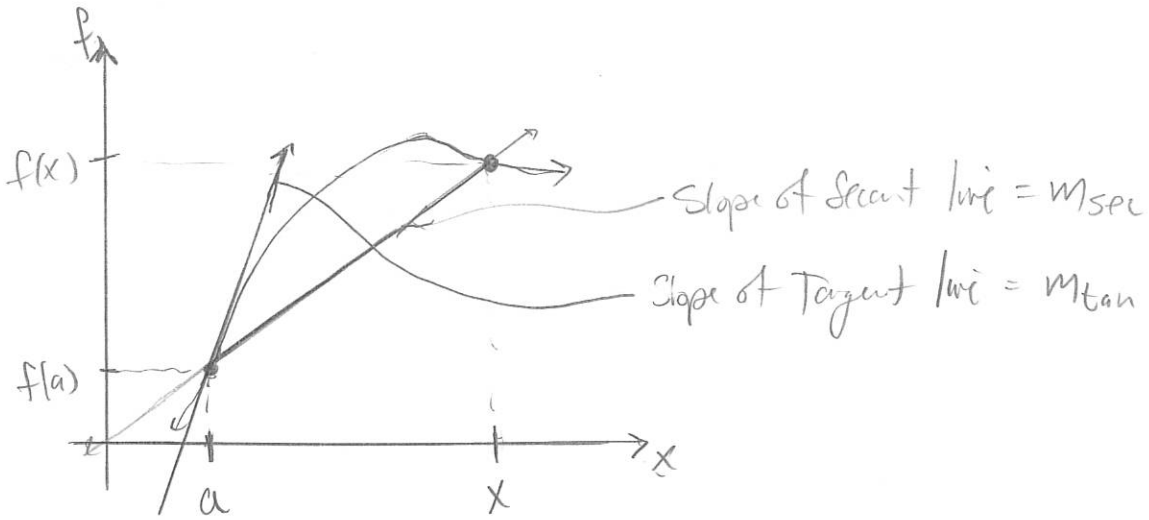
t_1 = start time

t_2 = end time

$f(x)$ = value of f at input value x

a = initial value

x = end value



Def: The average rate of change in f over the interval $[a, x]$ is the slope of the secant line:

$$m_{sec} = \frac{f(x) - f(a)}{x - a}$$

The instantaneous rate of change in f at $x = a$ is the slope of the tangent line at $(a, f(a))$ given by

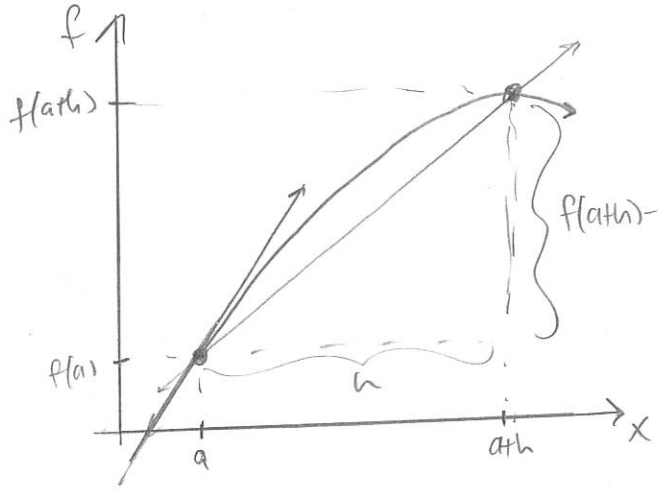
$$m_{tan} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

The equation of this line is unique and given by

$y - f(a) = m_{tan}(x-a)$ (Equation of tangent line at $x=a$)

WS #1, a, b, c. Practice finding secant line slope and tangent line slope.

- There is an alternate formula to the tangent line slope at $x=a$. Consider the following interval $[a, a+h]$ for $h>0$, then the distance between the two x-coordinates is always h:



$$m_{sec} = \frac{f(a+h)-f(a)}{h}$$

$$m_{tan} = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

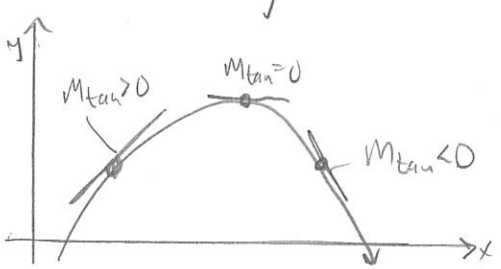
- In this definition, we see the following:

[Avg Rate of Change] = $\frac{f(a+h)-f(a)}{h}$

[Inst Rate of Change] = $\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$

WS #2 Instantaneous rate of change using alternative definitions.

- We have computed the slope of the tangent line at a single point on a curve. If this point varies along the curve, then the slope also varies along the curve.



- We define the slope of the tangent line of the function f at $x=a$ as $f'(a)$, i.e.

$$f'(a) = m_{tan} = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$$

- The slope of the tangent line of the function f is itself a function, called the derivative or derivative function, ASSUMING the limit above exists.

- So, we can create a new function, denote $f'(x)$, whose values are the slope of the tangent line of f at all values x where the following limit exists:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

← Domain of $f'(x)$ is all points x where this limit exists.

← Slope of TL (m_{tan}) or Instant Rate of Change of f at x.

Def: The derivative of f is the function

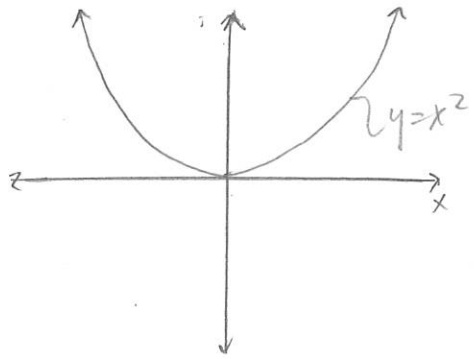
$$f'(x) = \lim_{x\to 0} \frac{f(x+h)-f(x)}{h}$$

provided the limit exists and x is in the domain of f. If $f'(x)$, we say f is differentiable at x. If f is differentiable at every point on an open interval, it is differentiable on that interval.

WS #3 Find $f'(x)$ using limit definition.

- The derivative function is a function whose input is any point x and the value $f'(x)$ is the slope of the function at the point $(x, f(x))$ on the curve.

Ex: Consider the graph of $f(x)=x^2$.



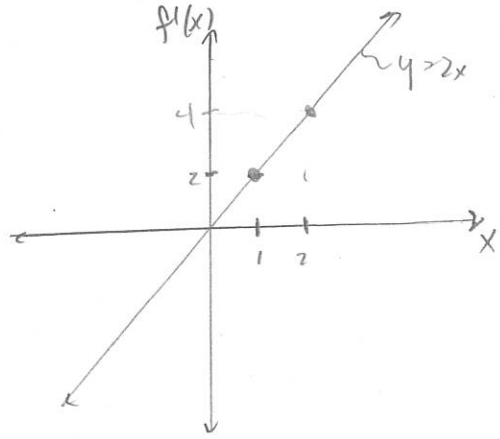
x	$f(x)$
-2	4
-1	1
0	0
1	1
2	4

↑ function values

x	$f'(x)$ = slope $= 2x$
-2	$f'(-2)<0 = -4$
-1	$f'(-1)<0 = -2$
0	$f'(0)=0 = 0$
1	$f'(1)>0 = 2$
2	$f'(2)>0 = 4$

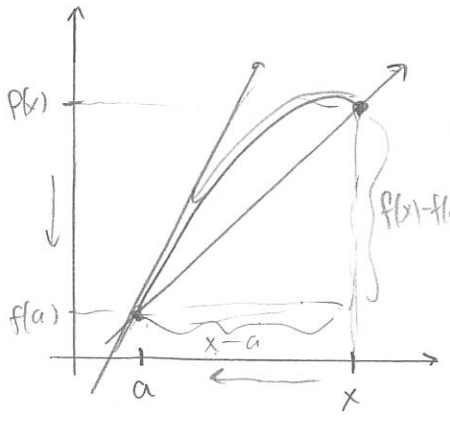
↑ slope values

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{(x+h)^2-x^2}{h}$$
$$=\lim_{h\to 0}\frac{x^2+2xh+h^2-x^2}{h}$$
$$=\lim_{h\to 0}2x+h$$
$$\boxed{f'(x)=2x}$$



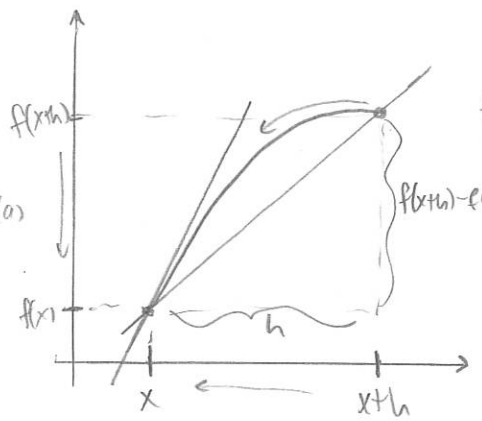
WS #1, a, b. Find and sketch $f'(x)$ using graph only.

- There is a lot of notation for the derivative:



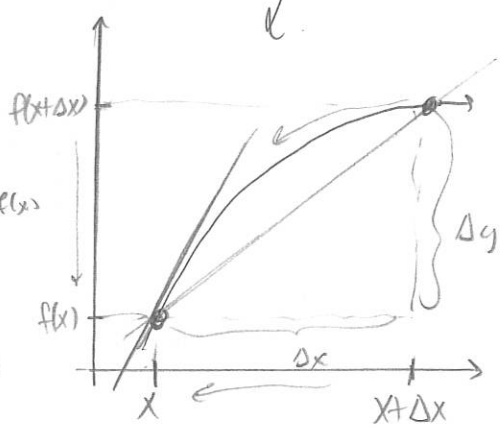
Change in y: $f(x)-f(a)$
Change in x: $x-a$

$$f'(a)=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$$



Change in y: $f(x+h)-f(x)$
Change in x: h

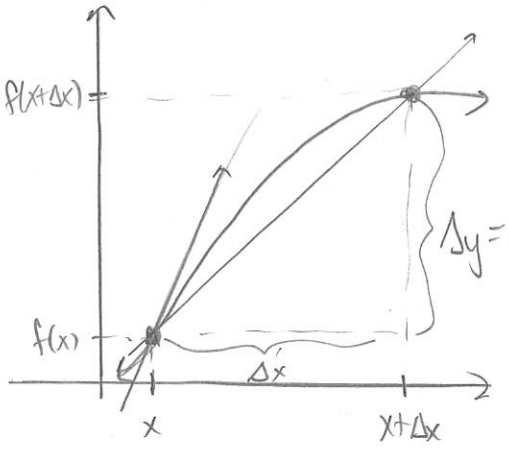
$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$



Change in y: $\Delta y = f(x+\Delta x)-f(x)$
Change in x: Δx

$$f'(x)=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\frac{dy}{dx}$$

- In the last case, we reserve the notation Δx to mean "change in x" and the term $\Delta y = f(x+\Delta x) - f(x)$ to mean "change in y". This gives



- $m_{sec} = \dfrac{\Delta y}{\Delta x} = \dfrac{f(x+\Delta x) - f(x)}{\Delta x}$

- $m_{tan} = \dfrac{dy}{dx} = \lim\limits_{\Delta x \to 0} \dfrac{\Delta y}{\Delta x} = \lim\limits_{\Delta x \to 0} \dfrac{f(x+\Delta x) - f(x)}{\Delta x}$

- Therefore, the derivative of the function $y = f(x)$ can be written in many ways:

$$\underbrace{f'(x),\ \frac{dy}{dx}}_{\text{Most commonly used}},\ \underbrace{y'(x), f', y', \frac{d}{dx}(f(x)), \frac{df}{dx}}_{\text{Used frequently}},\ \underbrace{D_x(f(x))}_{\text{Barely used}}$$

- Each of the following represent the derivative at the point $x = a$:

$$\underbrace{f'(a),\ \frac{dy}{dx}\Big|_{x=a}}_{\text{most commonly used}},\ \underbrace{y'(a),\ \frac{df}{dx}\Big|_{x=a},\ \frac{d}{dx}(f(x))\Big|_{x=a}}_{\text{Less commonly used.}}$$

- The prime notation belongs to Newton; $\frac{dy}{dx}$ is Leibniz notation.

WS #2 Find $g'(t)$ using limit definition.

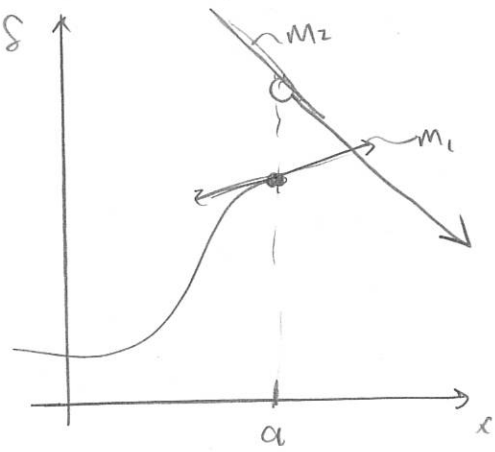
Finally, continuity has a special relationship with differentiability.

Thm: If f is differentiable at a, then f is continuous at a.

- The contrapositive of the statement is also true:

"If f is not continuous at a, then f is not differentiable at a"

- This statement may be easier to see:

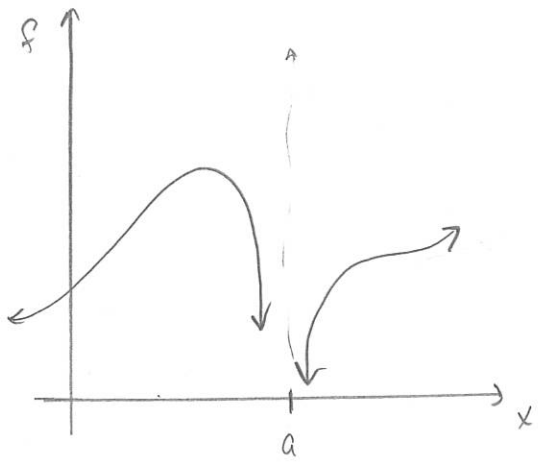


Obviously, f is not differentiable.

$m_1 \neq m_2$

$$m_1 = \lim_{x\to a^-} \frac{f(x)-f(a)}{x-a} \neq \lim_{x\to a^+} \frac{f(x)-f(a)}{x-a} = m_2$$

$$f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a} = \text{DNE}$$



f is not differentiable at a

$$\lim_{x\to a^-} \frac{f(x)-f(a)}{x-a} = -\infty \quad \lim_{x\to a} \frac{f(x)-f(a)}{x-a} = -\infty$$

$$\lim_{x\to a^+} \frac{f(x)-f(a)}{x-a} = -\infty$$

$\Rightarrow f'(a) = -\infty$?? (Can't happen!)

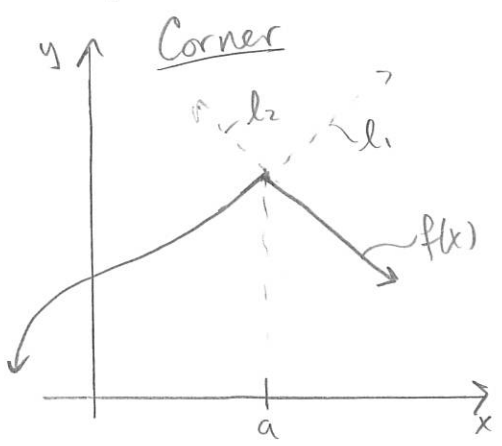
$f'(a) = \text{DNE}$.

- Hence, any discontinuities imply not differentiable.
- Can a function be continuous but not differentiable? Yes, we've already seen that when we differentiate a function, we "lose" parts of the domain!

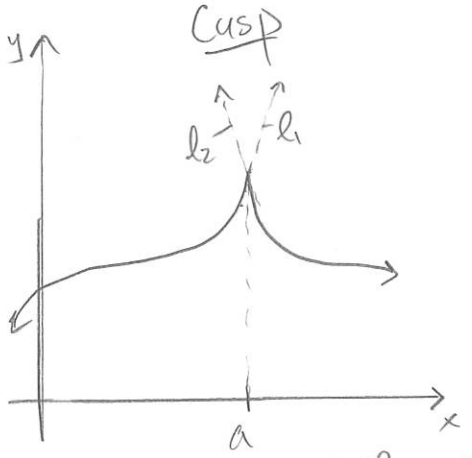
Ex: For the function $f(x) = \sqrt{x}$, we saw that $f(x)$ is continuous on $[0,\infty)$, right continuous at $x=0$ but it is not differentiable at $x=0$ since $f'(0) = \text{DNE}$. At that point, f has a vertical tangent line.

WS #3. Consider the function $f(x) = |x|$.

- In general, if the graph of f corrs to a corner or a cusp then f is continuous but not differentiable because the slopes do not "line up" at that point:



The slope of l_1 is different than the slope of l_2.



The slope of l_1 is different from the slope of l_2

This means $\lim_{h\to 0^-} \frac{f(a+h)-f(a)}{h} \neq \lim_{h\to 0^+} \frac{f(a+h)-f(a)}{h}$ which means

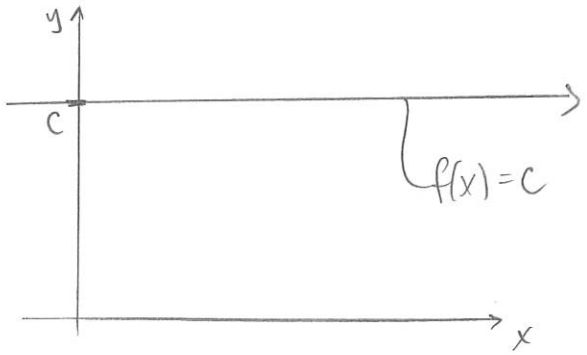
$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$ = DNE → Hence, $f'(a)$ = DNE

- In both cases above, $f(x)$ is continuous at $x=a$ but not differentiable at $x=a$.
- Conclusion: A function f is not differentiable at $x=a$ if at least one of the following conditions hold:
  1.) f is not continuous at $x=a$.
  2.) f has a vertical tangent at $x=a$.
  3.) f has a corner or cusp at $x=a$.

Section 3.2: Rules of Differentiation

- This section aims to make our lives easier by establishing rules for differentiation for particular types of functions.

- It is easy to see that the slope of a constant function is always zero:



Let $f(x) = c$, then

$m_{tan} = 0$

for all points x. Hence,

$$\boxed{\frac{d}{dx}(c) = 0} \text{ or } \boxed{f'(x) = 0}$$

Thm: (Constant Rule)

If c is a real number, then $\frac{d}{dx}(c) = 0$.

- Consider the function of the form $f(x) = x^n$, i.e. a power function. Then

$n = 1$: $\boxed{f(x) = x}$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$
$$= \lim_{h \to 0} \frac{x + h - x}{h}$$
$$= \lim_{h \to 0} \frac{h}{h}$$
$$= 1$$

$$\boxed{f'(x) = 1}$$

$$\boxed{f'(x) = 1x^0}$$

$n = 2$: $\boxed{f(x) = x^2}$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$
$$= \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}$$
$$= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{h}$$
$$= \lim_{h \to 0} \frac{h(2x+h)}{h}$$
$$= \lim_{h \to 0} 2x + h$$
$$= 2x$$

$$\boxed{f'(x) = 2x}$$

$$\boxed{f'(x) = 2x^1}$$

$n=3$: $\boxed{f(x)=x^3}$

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$
$$=\lim_{h\to 0}\frac{(x+h)^3-x^3}{h}$$
$$=\lim_{h\to 0}\frac{x^3+3x^2h+3xh^2+h^3-x^3}{h}$$
$$=\lim_{h\to 0}\frac{h(3x^2+3xh+h^2)}{h}$$
$$=\lim_{h\to 0}3x^2+3xh+h^2$$
$$=3x^2$$

$\Rightarrow \boxed{f'(x)=3x^2}$

So far, we have

n	$f(x)$	$f'(x)$
1	x^1	$1x^0$
2	x^2	$2x^1$
3	x^3	$3x^2$
⋮	⋮	⋮

$\boxed{f(x)=x^n \Rightarrow f'(x)=nx^{n-1}}$

Thm: (Power Rule)
If n is a positive integer, then $\frac{d}{dx}(x^n)=nx^{n-1}$.

Proof: Let $f(x)=x^n$. Then

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$
$$=\lim_{h\to 0}\frac{(x+h)^n-x^n}{h}$$
$$=\lim_{h\to 0}\frac{\sum_{i=0}^{n}\binom{n}{i}x^{n-i}h^i-x^n}{h}$$
$$=\lim_{h\to 0}\frac{x^n+\left(\sum_{i=1}^{n}\binom{n}{i}x^{n-i}h^i\right)-x^n}{h}$$
$$=\lim_{h\to 0}\frac{\sum_{i=1}^{n}\binom{n}{i}x^{n-i}h^i}{h}$$
$$=\lim_{h\to 0}\sum_{i=1}^{n}\binom{n}{i}x^{n-i}h^{i-1}$$
$$=nx^{n-1}$$

$\Rightarrow \boxed{f'(x)=nx^{n-1}}$

many rules that apply to limits follow through to the derivative since, of course, the derivative is nothing more than a limit.

Thm: (Constant Multiple Rule)

If f is differentiable and c is any constant, then

$$\frac{d}{dx}(cf(x)) = cf'(x)$$

Proof: we know $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ exists. Then we have

$$\begin{aligned}\frac{d}{dx}(cf(x)) &= \lim_{h\to 0} \frac{cf(x+h)-cf(x)}{h} \\ &= \lim_{h\to 0} \frac{c[f(x+h)-f(x)]}{h} \\ &= c\left(\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}\right) \\ &= cf'(x),\end{aligned}$$

as desired. ∎

- The sum/difference rule also applies.

Thm: (Sum/Difference Rule)

If f and g are differentiable, then

$$\frac{d}{dx}(f(x)+g(x)) = f'(x)+g'(x).$$

Proof: We know that $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ and $g'(x) = \lim_{h\to 0} \frac{g(x+h)-g(x)}{h}$ exist. Hence,

$$\frac{d}{dx}(f(x)+g(x)) = \lim_{h\to 0} \frac{(f(x+h)+g(x+h))-(f(x)+g(x))}{h}$$

$$= \lim_{h\to 0} \frac{(f(x+h)-f(x)) + (g(x+h)-g(x))}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} + \lim_{h\to 0} \frac{g(x+h)-g(x)}{h}$$

$$= f'(x) + g'(x)$$

as desired. ∎

- The fact that the power rule, constant multiple rule, and sum/difference rule can be applied, allows us to differentiate polynomials easily.

WS #1, a-d.

- Another important derivative is the derivative of the exponential function $f(x) = b^x$, where $b > 1$ is any real number.

- Note: x is in the exponent, not the base as with polynomials!
  Therefore, we cannot use the Power Rule:

$$f(x) = b^x \not\Rightarrow \underbrace{f'(x) = x\,b^{x-1}}_{\text{not Correct!}}$$

- In order to determine the derivative, we use the limit definition:

$$f(x) = b^x \quad \text{then} \quad f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{b^{x+h} - b^x}{h}$$

$$= \lim_{h\to 0} \frac{b^x(b^h - 1)}{h}$$

$$= \left(\lim_{h\to 0} \frac{b^h - 1}{h}\right) b^x$$

- We conclude that the derivative of b^x is just some constant times b^x:

$$f(x) = b^x \quad \text{then} \quad f'(x) = \underbrace{\left(\lim_{h\to 0} \frac{b^h - 1}{h}\right)}_{\text{Constant}} b^x$$

What is the value of this limit for each base b? Let $h = 0.0001$ and we'll investigate this derivative with Desmos:

Movie Use Desmos to graph the function $f'(x) = \left(\frac{b^{0.0001} - 1}{0.0001}\right) b^x$.

- We see that if $b = e$, the limit is exactly equal to 1! Hence, if $b = e$, we have

$$f(x) = e^x \quad \text{then} \quad f'(x) = e^x.$$

Def: (The number e) The number $2.71828182845\ldots$ satisfies

$$\lim_{h\to 0} \frac{e^h - 1}{h} = 1.$$

It is the base of the natural exponential function $f(x) = e^x$.

Thm: (The derivative of e^x). The function $f(x) = e^x$ is differentiable on $\mathbb{R}$, and

$$\frac{d}{dx}(e^x) = e^x.$$

- So e is the number such that

$$\lim_{h\to 0} \frac{e^h - 1}{h} = 1$$

but what is the value of this limit for any b:

$$\lim_{h\to 0} \frac{b^h - 1}{h} = ??$$ (If $b=e$, limit $= 1$)

If we knew this limit, we'd know the derivative of $f(x) = b^x$.

Movie Use desmos to graph $f(x) = \frac{x^{0.0001} - 1}{x}$.

- Hence, we see that for any b, we must have

$$\lim_{h\to 0} \frac{b^h - 1}{h} = \ln(b).$$

Thm: (derivative of b^x) The function $f(x) = b^x$ for $b > 1$ is differentiable on $\mathbb{R}$, and

$$\frac{d}{dx}(b^x) = \ln(b)\, b^x$$

- We can also find the derivative of a derivative.

Def: Assuming $f(x)$ can be differentiated as often as necessary, the second derivative of f is

$$f''(x) = f^{(2)}(x) = \frac{d^2 f}{dx^2} = \frac{d}{dx}(f'(x))$$

For integers $n \geq 1$, the nth derivative is

$$f^{(n)}(x) = \frac{d^n f}{dx^n} = \frac{d}{dx}(f^{(n-1)}(x))$$

WS #2 & #3, #4 work on derivatives

Section 3.3: The Product & Quotient Rules

As we saw in the previous section, limit laws (sum/diff and constant mult) can be applied directly to the derivatives. However, the product and quotient limit laws cannot be applied:

- $h(x) = f(x)g(x)$ then $h'(x) \neq f'(x)g'(x)$
- $k(x) = f(x)/g(x)$ then $k'(x) \neq \dfrac{f'(x)}{g'(x)}$

} Typical Limit laws do not apply here

Anti-Ex: Let $f(x) = x^3$ and $g(x) = x^2$, then $f'(x) = 3x^2$ and $g'(x) = 2x$. Consider

- $h(x) = f(x)g(x) = x^3x^2 = x^5 \Rightarrow h'(x) = 5x^4 \neq 6x^3 = f'(x)g'(x)$
- $k(x) = f(x)/g(x) = x^3/x^2 = x \Rightarrow k'(x) = 1 \neq \frac{3}{2}x = f'(x)/g'(x)$.

- So what are the product and quotient rules for derivatives?

Thm: (Product Rule) If f and g are differentiable, then

$$\frac{d}{dx}\big(f(x)g(x)\big) = f'(x)g(x) + f(x)g'(x).$$

Pf: Let f and g be differentiable, then

$$\begin{aligned}
\frac{d}{dx}\big(f(x)g(x)\big) &= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} \\
&= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h} \\
&= \lim_{h\to 0} g(x+h)\left(\frac{f(x+h)-f(x)}{h}\right) + \lim_{h\to 0} f(x)\left(\frac{g(x+h)-g(x)}{h}\right) \\
&= g(x)f'(x) + f(x)g'(x). \quad \blacksquare
\end{aligned}$$

LWS #3 work on other trig derivatives.

- As we continue to find more derivative rules, we see that new limits emerge:

$$\lim_{x\to 0}\frac{b^x-1}{x}=\ln(b), \qquad \lim_{x\to 0}\frac{\sin(x)}{x}=1, \qquad \lim_{x\to 0}\frac{\cos(x)-1}{x}=0$$

- We can continue to use these limits in the future

LWS #4 Working with limits

Section 3.5: Derivatives as Rates of Change

- We can use the idea of the derivative to model some real world applications. As we've seen before, the classic application of the derivative is the idea of velocity.

- One-Dimensional Motion:

  Assumption: motion of an object in one-dimension (eg. car moving along a straight highway, an object being launched vertically in the air.)

  Notation: t = time (hours)

  $s = s(t)$ = displacement from starting position (miles)
  $v = v(t)$ = velocity function after t hours (miles/hr)
  $a = a(t)$ = acceleration function after t hours (miles/hr^2)

- Average Velocity. The average velocity of the object over the interval $[a,b]$ is given by

$$[\text{Avg Vel}] = \frac{s(b)-s(a)}{b-a}$$

WS #3 a,-d. Work on the Product Rule

Next we have the quotient rule.

Thm: (Quotient Rule) If f and g are differentiable, then $\frac{d}{dx}(f/g)$ exists, provided $g(x) \neq 0$, and

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2} = \left[\frac{H_0 \, Dee \, H_1 - H_1 \, Dee \, H_0}{H_0 H_0}\right]$$

Proof: Let $q(x) = f(x)/g(x)$, then we wish to find $q'(x)$. we have

$$q(x) = f(x)/g(x)$$

$$\Rightarrow \quad f(x) = q(x)g(x).$$

Using the product rule, we have

$$f'(x) = q(x)g'(x) + q'(x)g(x).$$

we need to solve for $q'(x)$. we have

$$q'(x) = \frac{f'(x) - q(x)g'(x)}{g(x)}$$

$$\Rightarrow \quad q'(x) = \frac{f'(x) - \left(\frac{f(x)}{g(x)}\right)g'(x)}{g(x)}$$

$$\Rightarrow \quad q'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}$$

$$\Rightarrow \quad \frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$$

as desired. ∎

WS #4 a-d work on quotient rule.

- We can use the quotient rule to extend the power rule.

Thm: (Extended Power Rule) Let n be any integer, then

$$\frac{d}{dx}(x^n) = nx^{n-1}.$$

Proof: If $n>0$, we know this holds. Let $m=-n$, where $n>0$ is any positive integer. Then

$$\frac{d}{dx}(x^m) = \frac{d}{dx}(x^{-n})$$

$$= \frac{d}{dx}\left(\frac{1}{x^n}\right)$$

$$= \frac{x^n(0) - (1)(nx^{n-1})}{x^{2n}}$$

$$= -nx^{n-1}x^{-2n}$$

$$= -nx^{n-1-2n}$$

$$= -nx^{-n-1}$$

$$= mx^{m-1},$$

as desired. ∎

Thm: (Derivative of e^{kx}) Let k be any real number. Then

$$\frac{d}{dx}(e^{kx}) = ke^{kx}.$$

Proof: Let k be any real number then

$$e^{kx} = (e^k)^x$$

By the exponential rule, we have

$$\frac{d}{dx}(e^{kx}) = \frac{d}{dx}\left((e^k)^x\right)$$
$$= \ln(e^k)(e^k)^x$$
$$= ke^{kx}.$$

as desired.

WS #5 working on general derivatives

Section 3.4: Derivatives of Trigonometric Functions

- In this section, we consider the derivatives of $\sin(x)$ and $\cos(x)$.
- Consider $f(x) = \sin(x)$. Then

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$
$$= \lim_{h\to 0} \frac{\sin(x+h)-\sin(x)}{h}$$

$\sin(A+B) = \sin A\cos B + \cos(A)\sin(B)$

$$= \lim_{h\to 0} \frac{\sin(x)\cos(h) + \sin(h)\cos(x) - \sin(x)}{h}$$
$$= \lim_{h\to 0} \sin(x)\left(\frac{\cos(h)-1}{h}\right) + \lim_{h\to 0} \cos(x)\left(\frac{\sin(h)}{h}\right)$$
$$= \sin(x)\underbrace{\left[\lim_{h\to 0} \frac{\cos(h)-1}{h}\right]}_{?} + \cos(x)\underbrace{\left[\lim_{h\to 0} \frac{\sin(h)}{h}\right]}_{?}$$

- If we knew these two limits, we'd be done.
- Similarly, consider $f(x) = \cos(x)$:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\cos(x+h)-\cos(x)}{h}$$

$$= \lim_{h\to 0} \frac{\cos(x)\cos(h)-\sin(x)\sin(h)-\cos(x)}{h}$$

$$= \lim_{h\to 0} \cos(x)\left(\frac{\cos(h)-1}{h}\right) - \lim_{h\to 0} \sin(x)\left(\frac{\sin(h)}{h}\right)$$

$$= \cos(x)\underbrace{\left[\lim_{h\to 0} \frac{\cos(h)-1}{h}\right]}_{?} - \sin(x)\underbrace{\left[\lim_{h\to 0} \frac{\sin(h)}{h}\right]}_{?}$$

- The same two limits arise. Hence, we need to evaluate the following limits:

$$\lim_{x\to 0} \frac{\cos(x)-1}{x} = 0 \qquad \lim_{x\to 0} \frac{\sin(x)}{x} = 1$$

Desmos Use Desmos to show the limits above.

Thm: (Derivatives of sine and cosine)

$$\frac{d}{dx}(\sin(x)) = \cos(x) \qquad \frac{d}{dx}(\cos(x)) = -\sin(x)$$

WS #1 Practice these new derivatives

- Using these two derivatives to find the derivatives of all the other trig functions:

WS #2 Prove the derivative of $\tan(x)$ and $\csc(x)$.

Thm: (Derivatives of Trig functions)

$$\frac{d}{dx}(\sin(x)) = \cos(x) \qquad \frac{d}{dx}(\tan(x)) = \sec^2 x \qquad \frac{d}{dx}(\sec(x)) = \sec(x)\tan(x)$$

$$\frac{d}{dx}(\cos(x)) = -\sin(x) \qquad \frac{d}{dx}(\cot(x)) = -\csc^2 x \qquad \frac{d}{dx}(\csc(x)) = -\csc(x)\cot(x)$$

- Instantaneous Velocity: the instantaneous velocity, $v(t)$, at time t is given by the derivative of $s(t)$:

$$V(t) = s'(t) = \lim_{h \to 0} \frac{s(t+h) - s(t)}{h}$$

- Acceleration: the acceleration of the object at time t given by $a(t)$ is the second derivative of $s(t)$

$$a(t) = s''(t) = \lim_{h \to 0} \frac{v(t+h) - v(t)}{h}$$

WS #1 working with a position function

Note: Speed is mathematically defined as the absolute value of the velocity, i.e. speed is $|v(t)| = |s'(t)|$.

- A simple model of vertical motion can be obtained by assuming no air resistance and using Newton's second law:

Model: $F = ma \Rightarrow a = F/m$.

$F/m = -g$ on earth, which is $g = 9.8\ m/s^2$ or $g = 32\ ft/sec^2$

Hence

$$a = -32$$

$$\Rightarrow \boxed{s''(t) = -32}$$ This says that acceleration is constant on Earth.

This is called a differential equation. Can we solve for $s(t)$?

WS #2 working with vertical motion.

Growth Models: another application of the derivative deals with population growth or decay. Consider the following unlimited growth:

Assumption: the rate of change of the population is proportional to its current size.

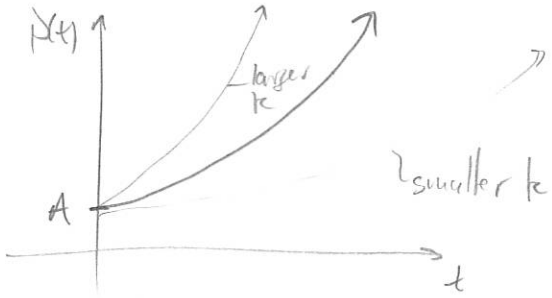
Notation: t = time (years)
P = population at time t (millions of people)
k = proportionality constant. ($k>0$, 1/years)

Model:

$$\frac{dP}{dt} = kP$$

- $\frac{dP}{dt}$: Rate of change
- kP: proportional to current size
- $\frac{dP}{dt} = kP$: Differential Equation for Pop. Growth.

Can we solve for $P(t)$? What function, $P(t)$, has a derivative that is a multiple of itself? $P(t) = Ae^{kt}$.



WS #3 Working with growth models

Section 3.6: The Chain Rule

- We now formulate a technique that can be used for composite functions.

- Consider the function $y = f(g(x))$. The "inside" function is $g(x)$ and the "outside" function is $f(x)$. Let $u = g(x)$, then $y = f(u)$. We see that f is a function of u and u is a function of x. This means

$$y = f(u) = f(g(x))$$

y is a function of x but only through composition with u. Hence, we differentiate in the following way:

$$\frac{dy}{dx} = \underbrace{\frac{dy}{du}}_{f'(u)} \cdot \underbrace{\frac{du}{dx}}_{u'(x)}$$

In substituting in for $u = g(x)$, we can write

$$\frac{dy}{dx} = \frac{d}{dx}\left(f(g(x))\right) = f'(g(x)) \cdot g'(x)$$

Essentially, we differentiate the outside function, leaving the inside function alone, then multiply by the derivative of the inside function.

Thm: (Chain Rule) Suppose $y = f(u)$ is differentiable at $u = g(x)$ where $g(x)$ is differentiable at x. The composite function $y = f(g(x))$ is then differentiable at x given by

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \quad \text{or} \quad \frac{d}{dx}\left(f(g(x))\right) = f'(g(x)) \cdot g'(x)$$

- In order to do the derivative of a composite function, it is important to identify the inside and outside functions.

WS #1 Identify the inside and outside functions.

- It is not necessary to always define the inner function as u. You can simply do the derivative directly.

WS #2 Using Chain Rule

- We can also find the slope at a single point using the chain rule.

WS #3 Derivative at a point.

WS #4 Chain Rule with more than 2 composite functions.

Section 3.7: Implicit Differentiation

- Typically, when we define a function of the variable x, we plot that function on the y-axis versus the x-axis by writing $y=f(x)$. This means y is explicitly defined in terms of x. Further, we know $y=f(x)$ passes the vertical line test.
- Not all relationships between two variables are defined explicitly. In fact, we know the following relationship maps a circle of radius $r=1$ in the x-y plane:

$$x^2+y^2=1$$

This relationship is defined implicitly in the variables x & y. We cannot actually represent this circle as a single function, but we can try:

$$x^2+y^2=1 \Rightarrow y^2=1-x^2 \begin{cases} \rightarrow y=\sqrt{1-x^2}=f_1(x) \\ \rightarrow y=-\sqrt{1-x^2}=f_2(x) \end{cases}$$

Here, we have that the two functions create the circle together

$y = f_1(x) = \sqrt{1-x^2}$ ← Top part of circle

$y = f_2(x) = -\sqrt{1-x^2}$ ← Bottom part of circle.

- If we wanted to find the slope of the curve, we could differentiate each function. However, this may not always be the case.
- Consider the relation

$$x + y^3 - xy = 1$$

This is very difficult to solve for y explicitly in terms of x. Hence, to find the slope, we consider implicit differentiation.

Ex: find the slope of the unit circle at the point $(1/2, \sqrt{3}/2)$.

$$x^2 + y^2 = 1$$

$$\Rightarrow \frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(1)$$

$$\Rightarrow \frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = 0$$

$$\Rightarrow 2x + 2y \cdot \frac{dy}{dx} = 0$$

↳ Must use chain rule when differentiating y with respect to x since y is dependent on x.

$$\Rightarrow 2y \frac{dy}{dx} = -2x$$

$$\Rightarrow \boxed{\frac{dy}{dx} = -x/y}$$ ← note that dy/dx depends on x & y

$$\Rightarrow \left.\frac{dy}{dx}\right|_{(1/2, \sqrt{3}/2)} = \frac{-1/2}{\sqrt{3}/2} = \boxed{-\frac{1}{\sqrt{3}}}.$$

- The key to these problems is to always remember that when differentiating y, an extra dy/dx will pop out due to the Chain Rule. We must do this since y is always defined as a function of x.

$$\frac{d}{dx}(x) = 1 \qquad \frac{d}{dx}(y) = y'$$

WS #1 Differentiating Implicitly.

WS #2 Find eqn of TL to an implicitly defined curve.

DESMOS Show the tangent line to curve.

- We can also find higher order derivatives of implicitly defined equations:

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}.$$

WS #3 Find second derivatives

- We can now prove the power rule for fractional exponents!

Thm: (Power Rule for Rational exponents)

Let p and q be integers such that $q \neq 0$. Then

$$\frac{d}{dx}\left(x^{p/q}\right) = \frac{p}{q}x^{p/q-1}.$$

Proof: Consider the function $y = x^{p/q}$, then

$$y^q = x^p \Rightarrow q y^{q-1}\cdot\frac{dy}{dx} = p x^{p-1}$$

$$\Rightarrow \frac{dy}{dx} = \frac{p}{q}\frac{x^{p-1}}{y^{q-1}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{p}{q}x^{p-1}y^{1-q}$$

$$\rightarrow \frac{dy}{dx} = \frac{p}{q}x^{p-1}\left(x^{p/q}\right)^{1-q}$$

$$\frac{dy}{dx} = \frac{p}{q}x^{p-1}x^{p/q-p}$$

$$\boxed{\frac{dy}{dx} = \frac{p}{q}x^{p/q-1}}$$

Section 3.8: Derivatives of Logarithmic Functions

- With the power of implicit differentiation, we can derive the derivative of logarithmic functions.

Thm: (Derivative of log functions)

Let $y = \log_b(x)$, then

$$\frac{d}{dx}\left(\log_b(x)\right) = \frac{1}{\ln(b)\,x} \qquad \left[\frac{d}{dx}\left(\ln(x)\right) = \frac{1}{x}\right]$$

Proof: Let $y = \log_b(x)$, then

$$y = \log_b(x)$$

$$\Rightarrow \quad b^y = x$$

diff $\Rightarrow \quad \frac{d}{dx}\left(b^y\right) = \frac{d}{dx}(x)$

$$\Rightarrow \quad b^y \ln(b)\frac{dy}{dx} = 1$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{\ln(b)\,b^y}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{\ln(b)\,x} \qquad \blacksquare$$

WS #1 a,b. Practice the derivative of logarithms

Section 3.9: Derivative of Inverse Trig Functions

- Using an identical method, we can find the derivatives of inverse trig functions.

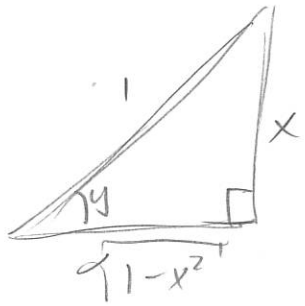
Thm: (Derivative of arcsin)

Let $y = \arcsin(x)$, then $\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$.

Proof: Let $y = \arcsin(x)$, then

$$y = \arcsin(x)$$
$$\Rightarrow \sin(y) = x$$
diff $$\Rightarrow \frac{d}{dx}(\sin(y)) = \frac{d}{dx}(x)$$
$$\Rightarrow \cos(y)\frac{dy}{dx} = 1$$
$$\Rightarrow \frac{dy}{dx} = \frac{1}{\cos(y)}$$

Since $\sin(y) = x = \frac{x}{1}$, we know we have the following right triangle:



Thus, $\cos(y) = \frac{\sqrt{1-x^2}}{1} = \sqrt{1-x^2}$. Hence,

$$\frac{d}{dx}(\arcsin(x)) = \frac{1}{\sqrt{1-x^2}}. \quad \blacksquare$$

WS #2 Prove the derivative of $\arctan(x)$.

Thm: (Derivatives of inverse trig functions)

$$\frac{d}{dx}(\arcsin(x)) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\arctan(x)) = -\frac{1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx}(\arccos(x)) = \frac{1}{1+x^2} \qquad \frac{d}{dx}(\operatorname{arccot}(x)) = -\frac{1}{1+x^2}$$

WS #3 Practice derivatives

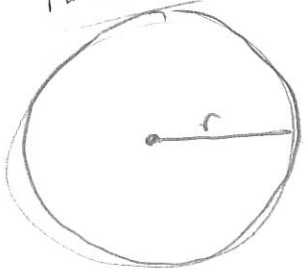
Section 3.10: Related Rates

- If two quantities that change over time are related to each other, then their rates of change are related to each other.

Ex: The area of a circle and the radius of that circle are related. In fact, if A = area and r = radius, then

$$A = \pi r^2$$

Suppose r is a constant value, then A is also constant.



Ex: $r = 5$, then

$$A = \pi r^2 = \pi(5)^2 = 25\pi$$

Here, since r is constant, we know $\frac{dr}{dt} = 0$. Hence

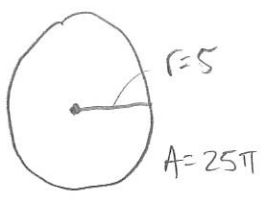
$$\frac{d}{dt}(A) = \frac{d}{dt}(\pi r^2) = \pi 2r \frac{dr}{dt} = 0$$

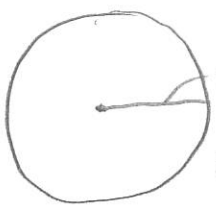
$$\Rightarrow \frac{dA}{dt} = 0$$

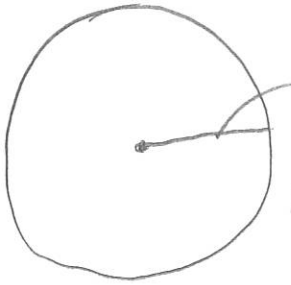
Therefore, the change of the area with time is zero.

Now, assume the radius is changing with time, so that $r = r(t)$. Assume $r(t) = (2t^3 + 5)$, then

$t = 0$: $r(0) = 2(0)^3 + 5 = 5$ $\quad$ $t = 1$: $r(1) = 2(1)^3 + 5 = 7$ $\quad$ $t = 2$: $r(2) = 2(2^3) + 5 = 21$







We wish to find a function that describes the rate of change of the area. Here, we know

$$r(t) = 2t^3 + 5 \quad \Rightarrow \quad A(t) = \pi(r(t))^2 = \pi(2t^3+5)^2$$

$$\Rightarrow \boxed{\frac{dr}{dt} = 6t^2} \text{ ft/sec} \qquad \Rightarrow \boxed{\frac{dA}{dt} = 2\pi(2t^3+5)(6t^3)} \text{ ft}^2\text{/sec}$$

Hence, we have a separate formula for dA/dt.

- Suppose we didn't know the actual function for $r(t)$, but we did know that the radius is increasing at 3 ft per sec. How fast is the area changing when the radius is 4 ft? Consider

$$\boxed{A = \pi r^2} \Rightarrow \frac{dA}{dt} = \frac{d}{dt}(\pi r^2) \quad \leftarrow \text{Implicit diff.}$$

$$= \pi \frac{d}{dt}(r^2)$$

$$\boxed{\frac{dA}{dt} = \pi 2r \frac{dr}{dt}}$$

↳ General relationship.

Hence, when $r = 4$ ft and when $\frac{dr}{dt} = 3$, we have

$$\frac{dA}{dt} = \pi 2(3)(4) = \boxed{24\pi} \text{ ft}^2\text{/sec}.$$

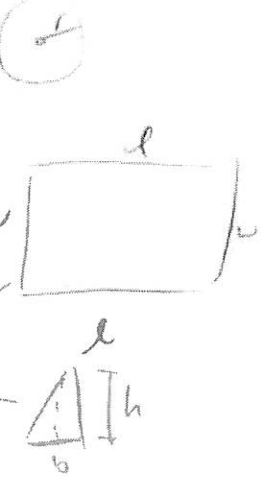
we can still relate the rates of change to find $\frac{dA}{dt}$.

WS #1 working with a cube.

- Many related rates problems involve two rates, one of which is known/given and one in which you must find.
- In general, we may have to recall specific formulas that relate the variables of interest: 2-D and 3-D formulas for area, surface area, or volume, Pythagorean Theorem & Similar triangles. WS #2-#4.

- Many related rate problems involve two rates, one of which is known and one in which you must find.
- Many related rate problems involve geometric quantities such as volume, area, or surface area.

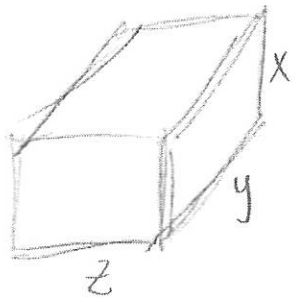
- 2-D Formulas:
  - Area of Circle: $A = \pi r^2$
  - Circumference: $C = 2\pi r$
  - Area of Rectangle: $A = lw$
  - Perimeter of Rectangle: $P = 2l + 2w$
  - Area of Triangle: $A = \frac{1}{2}bh$



- 3-D Formulas:
  - Rectangular Box:

$V = xyz$

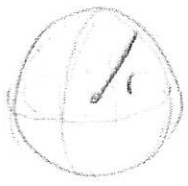
$SA = 2xy + 2yz + 2xy$



- Sphere:

$V = \frac{4}{3}\pi r^3$

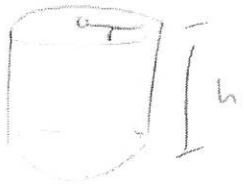
$SA = 4\pi r^2$



- Cylinder:

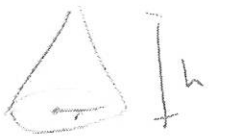
$V = \pi r^2 h$

$SA = 2\pi r^2 + 2\pi rh$



- Cone:

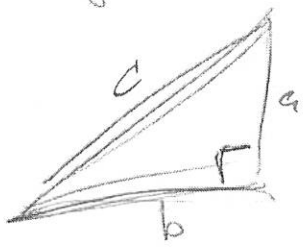
$V = \frac{1}{3}\pi r^2 h$



- It is also common for related rate questions to involve the well-known theorems for right triangles.

- Pythagorean Theorem:

$$c^2 = a^2 + b^2$$

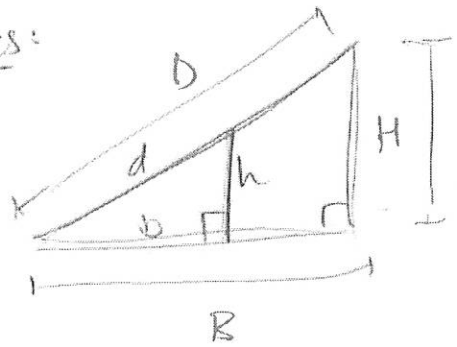


- Similar Triangles:

$$\frac{d}{b} = \frac{D}{B}$$

$$\frac{b}{h} = \frac{B}{h}$$

$$\frac{d}{h} = \frac{D}{H} \quad \ldots$$



Ex WS

Ex WS

Section 3.6: L'Hopital's Rule

- As we've seen already seen, there exists limits of the form $\frac{0}{0}$ that are considered indeterminate.
- It is not always clear if limits of this form exist.

Recall: $\lim_{x \to 0} \frac{x^2}{x^3 - x} = \lim_{x \to 0} \frac{x}{x^2 - 1} = \frac{0}{0^2 - 1} = 0$

or: $\lim_{x \to 0} \frac{x^2}{2^x - 1} = \frac{0}{0}$ ← Algebra doesn't work