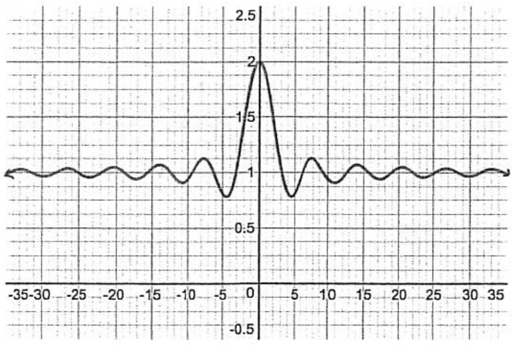
§2.5: Limits at Infinity

A function is allowed to cross its HA

1.] Evaluate $\lim_{x\to\infty} 1 + \frac{\sin(x)}{x}$. Does this function have a horizontal asymptote?



Yes, because as $x \to \infty$ and as $x \to -\infty$, the function is getting closer to 1 even though it "bounces" around that value.

$$\lim_{x\to\infty} 1 + \frac{\sin(x)}{x} = 1 + \lim_{x\to\infty} \frac{\sin(x)}{x} = 1 + \underbrace{\frac{(\text{something between } -1 \,\&\, 1)}{\infty}}_{\text{goes to } 0}$$

$$\lim_{x\to-\infty} 1 + \frac{\sin(x)}{x} = 1 + \frac{(\text{something between } 1 \,\&\, -1)}{-\infty} = \boxed{1}$$

2.] Determine the limits at infinity for the following polynomial: $f(x) = 4x^4 - 2x^3 + x - 100$

$$\lim_{x\to\infty} 4x^4 - 2x^3 + x - 100 = \lim_{x\to\infty} x^4\left(4 - \frac{2}{x} + \frac{1}{x^3} - \frac{100}{x^4}\right) = (\infty)^4(4-0+0-0) = (\infty)(4) = \boxed{\infty}$$

$$\lim_{x\to-\infty} 4x^4 - 2x^3 + x - 100 = \lim_{x\to-\infty} x^4\left(4 - \frac{2}{x} + \frac{1}{x^3} - \frac{100}{x^4}\right) = (-\infty)^4(4-0+0-0) = (\infty)(4) = \boxed{\infty}$$

3.] Determine the end behavior of the following rational functions:

a.) $f(x) = \dfrac{2x+1}{x^2-1}$

$$\lim_{x\to\infty} \frac{2x+1}{x^2-1} = \lim_{x\to\infty} \frac{x(2+1/x)}{x^2(1-1/x^2)} = \lim_{x\to\infty} \frac{2+\frac{1}{x}}{x(1-1/x^2)} = \frac{2}{(\infty)(1)} = \boxed{0}$$

$$\lim_{x\to-1} \frac{2x+1}{x^2-1} = \lim_{x\to-\infty} \frac{x(2+1/x)}{x^2(1-1/x^2)} = \lim_{x\to-\infty} \frac{2+\frac{1}{x}}{x(1-1/x^2)} = \frac{2}{(-\infty)(1)} = \boxed{0}$$

HA at $y=0$.

b.) $g(x) = \dfrac{20x^4 - 6x^2 - 10}{4x^4 + 8x^2 + 1}$

$$\lim_{x\to\infty} \frac{20x^4-6x^2-10}{4x^4+8x^2+1} = \lim_{x\to\infty} \frac{20 - \frac{6}{x^2} - \frac{10}{x^4}}{4 + \frac{8}{x^2} + \frac{1}{x^4}} = \frac{20}{4} = \boxed{5}$$

$$\lim_{x\to-\infty} \frac{20x^4-6x^2-10}{4x^4+8x^2+1} = \lim_{x\to-\infty} \frac{20 - 6/x^2 - 10/x^4}{4 + 8/x^2 + 1/x^4} = \frac{20}{4} = \boxed{5}$$

HA at $y=5$

c.) $h(x) = \dfrac{-x^3 + x - 5}{3x+1}$

$$\lim_{x\to\infty} \frac{-x^3+x-5}{3x+1} = \lim_{x\to\infty} \frac{x^3(-1+1/x^2-5/x^3)}{x(3+1/x)} = \lim_{x\to\infty} \frac{x^2(-1+1/x^2-5/x^3)}{3+1/x} = \frac{\infty^2(-1)}{3} = \boxed{-\infty}$$

$$\lim_{x\to-\infty} \frac{-x^3+x-5}{3x+1} = \lim_{x\to-\infty} \frac{x^3(-1+1/x^2-5/x^3)}{x(3+1/x)} = \lim_{x\to-\infty} \frac{x^2(-1+1/x^2-5/x^3)}{3+1/x} = \frac{(-\infty)^2(-1)}{3} = \boxed{-\infty}$$

No HA

4.] Determine the end behavior of the following algebraic function:

$$f(x) = \frac{4x^3+1}{\sqrt{16x^6+x^2-1}}$$

Notice that $f(x)$ can be written as

$$f(x) = \frac{x^3(4+1/x^3)}{\sqrt{x^6(16+1/x^4-1/x^6)}} = \frac{x^3(4+1/x^3)}{\sqrt{x^6}\sqrt{16+1/x^4-1/x^6}}$$

Also, note $\sqrt{x^6} = x^3$ for $x>0$ and $\sqrt{x^6} = -x^3$ for $x<0$. We have

$\underline{x>0:}$ $$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x^3(4+1/x^3)}{x^3\sqrt{16+1/x^4-1/x^6}} = \lim_{x\to\infty} \frac{4+1/x^3}{\sqrt{16+1/x^4-1/x^6}} = \frac{4}{\sqrt{16}} = \boxed{1}$$

$\underline{x<0:}$ $$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{x^3(4+1/x^3)}{-x^3\sqrt{16+1/x^4-1/x^6}} = \lim_{x\to\infty} \frac{4+1/x^3}{(-1)\sqrt{16+1/x^4-1/x^6}} = \frac{4}{-\sqrt{16}} = \boxed{-1}$$

This function has two HA's: $y=1$ and $y=-1$.

5.] Determine the end behavior of the following transcendental functions:

a.) $f(x) = -3e^{-x}$

$$\lim_{x\to\infty} -3e^{-x} = -3e^{-\infty} = \frac{-3}{e^{\infty}} = \boxed{0}$$ HA at $y=0$

$$\lim_{x\to-\infty} -3e^{-x} = -3e^{\infty} = -3(\infty) = \boxed{-\infty}$$

b.) $g(x) = \ln\left(\frac{1}{x}\right)$

$$\lim_{x\to\infty} \ln\left(\tfrac{1}{x}\right) = \lim_{x\to\infty} \ln(1) - \ln(x) = \lim_{x\to\infty} 0 - \ln(x) = \lim_{x\to\infty} -\ln(x) = \boxed{-\infty}$$

$$\lim_{x\to-\infty} \ln\left(\tfrac{1}{x}\right) = \boxed{\text{DNE}}$$ (out of domain)

c.) $h(x) = \sin(x)$

$$\lim_{x\to\pm\infty} \sin(x) = \boxed{\text{DNE}}$$ seize